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SOUND AND STRUCTURAL VIBRATION

by Preston W. Smith, Jr.,
and Richard H. Lyon

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I. INTRODUCTION

This report is designed to be a systematic development of some new techniques for analyzing structural vibrations and the interactions between sound fields and structural vibrations. In its application to structural vibration, the approach is quite new, having been motivated by the lengthening roster of difficult questions concerning vibrations in very complicated structures -- buildings, missiles, ocean vessels, etc. -- which are caused by a complicated set of forces. The new techniques rely upon an old trick: to make a "difficult" problem "easy", ask the easy questions. It is astounding how often the answers to easy questions will suffice.

In the classical approach to a vibration problem, one usually asks, "What is the dynamic displacement of a particular point at a particular instant?" Now, in many practical problems, this is a most unreasonable question. As with the question "What is the present population of China?", no reasonable effort can yield an answer. Even if an answer were forthcoming, from that ideal computer that analysts dream of, it would not be useful because <u>particular</u> points and <u>particular</u> instants are not really of concern, and a collection of data for <u>all</u> points and <u>all</u> instants would be overwhelming.

To get a useful answer, some different question must be posed; let us try, "What is the average dynamic response (in a root-mean-square sense) when that average is performed both in space and in time?" This is better; at least the answer is one handy number. However, too much information has been lost in the process. For example, the answer says nothing about the time rate of change of response (i.e., about the frequencies involved), and such information is often important.

The nature of the problem and some idea of the answers desired can be brought out by describing a typical practical situation. A very large rocket carries a moderately large capsule inside of which are mounted, in various ways and positions, some packages of delicate electronic instruments. Too much vibration of any one of many vacuum tubes, for example, will cause the whole rocket to misbehave. It is thought that the vibration may be caused by sound from the rocket engine passing through the capsule, reverberating about inside, and forcing the package of electronics. An estimate of the vibration generated in this manner is desired so that possible protective modifications to the structure and the instruments can be evaluated in a rational manner.

The sound inside the capsule is found to be an extremely complicated function of time. It is a more or less random noise, although the energy is not distributed uniformly in frequency. Moreover, because of the limited space in the capsule, sound does not reach the package from any single direction. The sound bounces around in the space, and is repeatedly directed from many different angles, a fate which markedly affects the spatial distribution of force on the package.

The vibratory response of the package to sound waves can be studied in the laboratory, irradiating it with a pure-tone wave incident from a single direction. One will then typically find that the response at a single point fluctuates tremendously as frequency is varied, being very large in small regions of frequency near the natural mechanical resonances of the package. Figure I.1 is an example of the records that are obtained in tests of this sort. At any one of these natural frequencies, the response may vary quite considerably, depending upon the angle from which the sound wave arrives. Finally, the magnitude of response varies from point to point, when frequency and angle of incidence are held fixed.

Upon inquiry, one discovers that the various electronic elements are sensitive to vibration in various ranges of frequency, and that their exact locations either are not known, are subject to change, or are distributed widely throughout the package. It is now evident that no exact question can be posed; it is needless to search for exact answers, of the type we called "classical". Only some sort of average, statistical estimates of response are required.

In this report we shall develop analytical procedures for obtaining estimates of this sort. In crude outline, the procedures more or less parallel the experimental laboratory study just described. From design drawings, one estimates the average number of resonances expected in a moderately broad band of frequency and the spatial distribution of response amplitude for a "typical" mode of resonance. With this information, one estimates the average response of a single "typical" mode to sound waves of noise incident from many various angles. The product of this average response per resonant mode by the average number of modes in a frequency band yields an estimate for the space-time average response in that frequency band. The process is repeated for different bands.

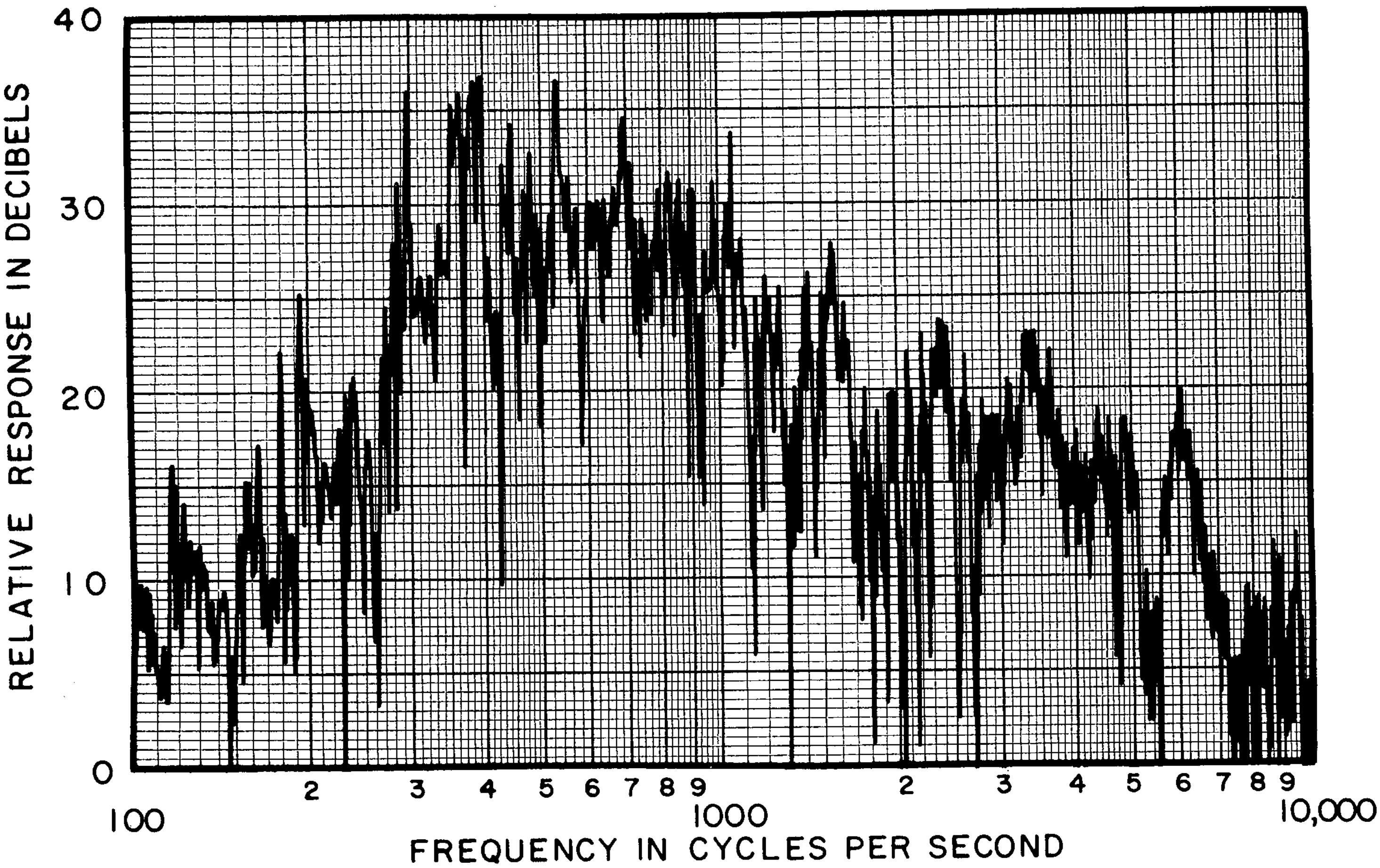

Figure I.1.- Vibratory acceleration at one point of an aluminum panel exposed to a pure-tone sound wave of constant pressure and slowly varied frequency (20 db corresponds to a factor 10 in response).

It is, of course, hopeless to attempt to find the exact characteristics of individual modes in the early phases of such an analysis. The saving feature of this new approach is that many of the average characteristics -- number of modes in a frequency band, average coupling to sound incident from various angles, etc. -- are insensitive to those details of construction which render impossible the exact analysis.

The aims of the present study are three-fold: (i) To outline a conceptual framework for analyzing the vibrations of complex distributed systems and the interactions of several systems. The approach proceeds by (a) a multi-modal description of the vibrations of a system; (b) characterization of the modal response by the vibratory energy involved, and of the interactions by energy flux; (c) formulation of statistical average estimates of the dynamic parameters of various systems. (ii) To incorporate the principal results of earlier studies of vibratory interactions between sound fields and bending vibration fields in panel-like structures. (iii) To establish a unified basis for continuing research and extensions to new problems.

It is anticipated that many readers of this report will be non-specialists in one or another of the pertinent technical fields: acoustics, mechanical vibrations, and statistics. For this reason, the report develops the necessary concepts of each field from principles so fundamental as to bore the specialist. However, some familiarity with calculus and vectors is presumed.

The simple resonator, consisting of a mass, a spring, and a dissipative mechanism, is the analogue for response in a single natural mode of a complicated structure. Chapter II is a study of the response of a single resonator and of sets of resonators. These mechanically simple problems furnish a ready opportunity to introduce many of the fundamental concepts: spectral analysis, mechanical impedance and its relation to energy, frequency-dependent coupling parameters, modal density, ensemble averages, etc.

Sound waves are discussed in Chapter III. From basic principles, the treatment proceeds to those concepts later required for analyzing the coupling between sound and structures: acoustical energetics, sound generation and radiation impedances, wavenumber vector and trace wavenumber, room acoustics, modal density, reverberation, and diffuse fields.

Structural vibrations and waves are presented in Chapter IV. Only bending (flexural) motion is considered, since practical response problems in extended structures are associated almost exclusively with bending. The treatment closely parallels that of sound waves in the previous chapter. An example of coupled mechanical systems is discussed.

Sound waves and structural vibrations are finally brought together in Chapter V. Here, the method for modelling structural vibration by the oscillations of a set of simple resonators is considered in detail. The concepts of directivity and reciprocity are introduced. From results in Chapter II, formulas are derived for the one-mode and multi-modal response of a general structure to noise and diffuse sound fields.

Chapter VI considers the evaluation of coupling between sound waves and bending vibrations of flat beams and panels. Formulas appropriate to numerical prediction are given. Comparisons between experiments and theory are shown.

As a general technique, the procedures outlined in this report are in mid-evolution. First results are but a few years old; refinements and extensions are matters of current research. Chapter VII contains a survey of the literature and current research, for the guidance of interested readers.

II. THE SIMPLE RESONATOR

II.1 Introduction

Repeatedly in this text we shall require knowledge of various characteristics of the response of the simplest resonating system: a spring-mounted mass, with a "dashpot" in which the energy of motion is dissipated. Figure II.1 shows a conventional sketch of the simple resonator, with the symbols and nomenclature that will be used.*

Of course, this particular system is seldom met in actual vibration problems, but it serves admirably as analogue for the motion of more complicated structures. Specifically, the amplitude of response of any one natural mode of a complicated structure can be identified with the response of a suitably constructed simple resonator. The total response of the complicated structure can be compounded from the responses of a set of resonators corresponding to all its natural modes. This process is the basis of the present study. The analytical procedures can all be brought out in a discussion of simple resonators. This chapter is such a discussion, starting with a single resonator and ending with many. But, application of the procedures to practical problems requires the ability to calculate, for example, the "external force f(t)" which corresponds to a specified sound wave. These practical matters will be taken up in later chapters.

The simple resonator is a classical problem, and detailed derivations of its response will be found in numerous introductory texts on mechanical and electrical oscillations.** This chapter presumes a prior acquaintance with these details, but not a thorough knowledge. The results will be set out here without elaborate proofs, to serve as a ready reference for later applications and as a convenient place to define standard notation and symbols. The latter parts of the chapter present aspects of the response not found in introductory texts, and discuss the generalization to more complicated structures.

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* Figures are numbered consecutively within each chapter and are located at the end of each chapter's text.

**For a few examples, see J. P. Den Hartog, <u>Mechanical Vibrations</u>, 4th Ed. (McGraw-Hill Book Company, New York, 1956); S. H. Crandall <u>Random Vibrations</u>, Vol. 1 (The Massachusetts Institute of Technology Press, Cambridge, 1959), Chapters 1 and 4.

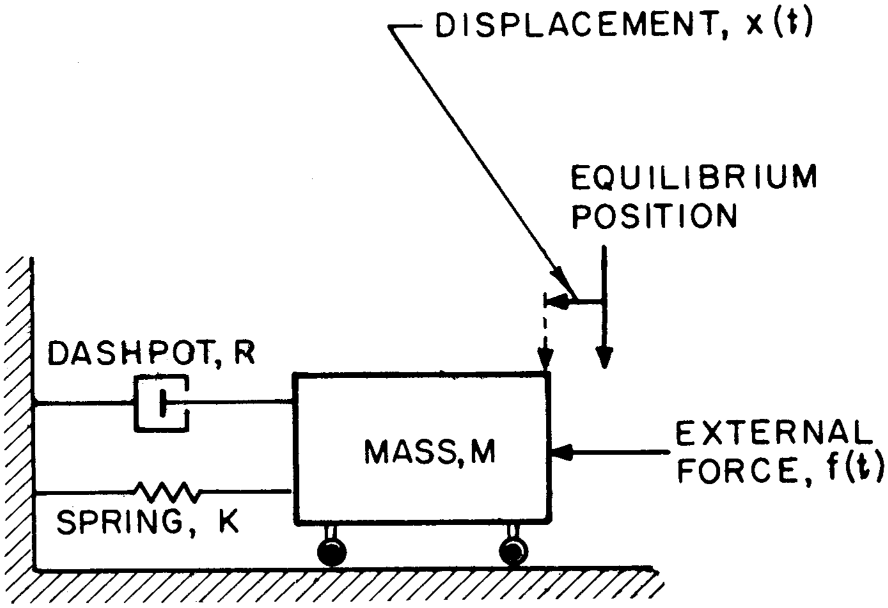

Figure II.1.- The simple resonator.

Throughout, we consider only linear vibrations, involving small displacements from the position of static equilibrium of systems whose dynamical characteristics do not change in time.

II.2 Equations of Motion; Energy Functions

The prototype of simple resonators is a structure consisting of a mass M mounted on an ideal spring of stiffness K, and also connected to a "dashpot" which resists motion by exerting a force proportional to velocity (Fig. II.1). The position of the mass at any instant t is specified by its displacement $x(t)$ from the position of static equilibrium. In addition to the forces between these parts of the system, there are other, external forces $f(t)$ that act on the mass.

The differential equation governing the instantaneous displacement $x(t)$ equates the total force acting on the mass with the product of mass and acceleration:

$$M\ddot{x}(t) = -R\dot{x}(t) - Kx(t) + f(t) \qquad \text{(II.2.1)*}$$

where the terms on the right are, successively the resistive force of the dashpot, the spring force, and the external force $f(t)$. The superscript dot indicates the time derivative; hence $\dot{x}(t) = v(t)$ is the velocity of the mass. The coefficient R is called the resistance.

There are three energy functions associated with vibration of the resonator: the kinetic energy T of the mass; the potential energy U stored in the spring; Rayleigh's dissipation function D which is a measure of the rate at which the dashpot dissipates vibratory energy, i.e. converts it to other forms of energy, such as heat**. The expressions for the first two are well known:

$$\begin{aligned} T &\equiv \tfrac{1}{2} M\, v^2(t) \\ U &\equiv \tfrac{1}{2} K\, x^2(t) \quad . \end{aligned} \qquad \text{(II.2.2)}$$

* All equation numbers have three parts which indicate, respectively, the chapter, the section within the chapter, the equation within the section.

**Elsewhere the dissipation function is almost universally denoted by a capital F, either roman or script. We use D to avoid confusion with symbols for force.

These will sometimes be met in combination. Their sum $E \equiv T + U$ is the total energy of vibration. Their difference $L \equiv T - U$ is called the Lagrangian of the system.

The dissipation function is defined as half of the instantaneous rate at which vibratory energy is lost from the system. This rate of loss of energy is also called the power dissipated, and is denoted by Π. The analytical expressions for the dissipation function and the dissipated power are given by

$$2D \equiv \Pi \equiv (-f_R)v = R\, v^2(t) \quad . \qquad \text{(II.2.3)}$$

Note that it is not at all essential that the dashpot convert vibratory energy into heat. Any mechanism whereby part of the energy E is abstracted from the resonator, at a rate $\Pi = 2D$ proportional to $v^2(t)$, is perfectly well described by the dashpot. Indeed, we shall often be concerned with "dissipation" in which the energy is carried out of the resonator in the form of sound waves or as stress waves in other parts of the whole structure.

The differential equation of motion can be written in terms of the energy functions instead of the parameters M, R, K. Lagrange's equation for the forced vibration of the simple resonator is

$$\frac{d}{dt}\frac{dT}{dv} - \frac{dV}{dx} + \frac{dD}{dv} = f(t) \quad . \qquad \text{(II.2.4)}$$

It is readily verified that this is identical with Eq. II.2.1.

II.3 Averaging Notation

We digress from the course of analysis to specify some conventions of notation. As a practical matter, one is seldom interested in instantaneous values of force, velocity, etc. The force and the response will generally be characterized instead by various time average values. There is a simple reason for this. The time average values are single numbers which in many situations contain all the information necessary to make engineering decisions.

As this study progresses, we shall also find it necessary to take averages over variables other than time. For example, we may look for the average over position on the surface of a panel of the time-average of the square of vibratory displacement; the time-average value is different at different points on the surface. Or, having found the response of a structure to excitation by a sound wave incident from a particular direction, we may look for the average response for all possible angles of incidence. Thus we often find ourselves averaging over three or four different variables at the same time!

In such a situation, it is very easy for notation to conceal common sense and meaning. It is very hard to find a notation that always avoids the turgidity of elaborate precision, on the one hand, and the confusion of ambiguous simplicity on the other. A compromise has been adopted for this study.

Angular brackets, $\langle\ \rangle$, will denote the simple (i.e., unweighted) average of the function enclosed. For example, the average over all time of the square of velocity $v(t)$ is

$$\langle v^2(t)\rangle = \lim_{\tau\to\infty}\left[\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau} v^2(t')dt' \Big/ \int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau} dt'\right] \quad . \qquad \text{(II.3.1)}$$

We shall call this average the mean-square velocity, and its square root the root-mean-square or rms velocity, but we shall not adopt any special notation for it. It may be abbreviated to $\langle v^2\rangle$ where no ambiguity is involved.

One must be more explicit when the function depends on several variables. If $f(x,t)$ is a function of position x as well as time t, then the time-average will usually be denoted $\langle f(x,t)\rangle_t$. It is still a function of position. The space-average $\langle f(x,t)\rangle_x$ remains a function of time. The space-time average is written $\langle f(x,t)\rangle_{x,t}$. Where the subscripts seem superfluous, they may be omitted in favor of an appropriate statement in the text.

Sometimes an average will be taken with respect to a discrete index number. Thus, if M_α is the mass of the α-th structural member, where the index number α takes on values 1,2,3, ..., then $\langle M_\alpha\rangle_\alpha$ denotes the average mass of a member. (The number of members may be either finite or infinite.)

Often the averages do not extend over all possible values of the variable. For example, suppose a structural beam is oriented along the x-axis between the points x=0 and x=L. Let f(x) stand for the load (force per unit length) at a general point x. When it is necessary or desirable to be explicit, we shall add some explanatory subscript to indicate the limited range of variable, as in these examples:

$$\langle f(x)\rangle_{0<x<L} \qquad\qquad \langle f(x)\rangle_{\text{on beam}}.$$

II.4 Natural Vibrations

Vibrations of a resonator in the absence of external forces are called natural. Consider first a resonator with no damping. The displacement must satisfy the differential equation

$$M\ddot{x}(t) + Kx(t) = 0 \quad .$$

The general solutions for velocity v(t) and displacement x(t) represent steady sinusoidal motion with a single frequency:

$$\begin{aligned} v(t) &= V\cos(\omega_o t + \phi) \\ x(t) &= (V/\omega_o)\sin(\omega_o t + \phi) \\ \omega_o^2 &\equiv K/M \end{aligned} \qquad \text{(II.4.1)}$$

where V and ϕ are arbitrary constants that can be determined from initial conditions. The natural angular frequency ω_o has the units radians per second; the frequency in cycles per second is $(\omega_o/2\pi)$.

The time-averaged, or mean, displacement and velocity both vanish, of course. The mean square values do not. The average of the square of a sine wave over an interval τ which is any integral number of half-periods, π/ω_o sec, is exactly 1/2; this is also the long-time average value (i.e., the value of Eq. II.2.1 in the limit as $\tau\rightarrow\infty$). Thus, for the undamped natural vibration of the simple resonator, we have mean square values

$$\begin{aligned} \langle v^2\rangle &= \tfrac{1}{2}V^2 \\ \langle x^2\rangle &= \tfrac{1}{2}V^2/\omega_o^2 \quad . \end{aligned}$$

Both the kinetic and potential energies fluctuate about their average values, but their sum E is absolutely constant; the form of the energy changes but the total amount does not. Moreover, the average values of T and U are equal, so that the average Lagrangian vanishes. The principal analytical relations for the energy functions of an undamped resonator are:

$$\begin{aligned}\langle T\rangle &= \frac{1}{2} M\langle v^2\rangle = \frac{1}{4} M V^2 \\ \langle U\rangle &= \frac{1}{2} K\langle x^2\rangle = \frac{1}{4} K V^2/\omega_o^2 = \langle T\rangle \\ \langle L\rangle &= \langle T-U\rangle = 0 \\ E &= T + U = \langle T+U\rangle = 2\langle T\rangle = M\langle v^2\rangle \quad .\end{aligned} \qquad (II.4.2)$$

In the presence of damping, the total energy of natural vibrations decreases steadily with time. Throughout, we shall be concerned with small damping, in which case the natural vibration is very closely approximated by a sinusoidal oscillation at the undamped natural frequency ω_o, with an amplitude that decays exponentially with time. A measure of damping is the <u>loss factor</u>

$$\eta \equiv R/\omega_o M = R\omega_o/K \quad . \qquad (II.4.3)$$

The criterion for "small damping" is

$$\left(\frac{1}{2}\eta\right)^2 << 1 \quad .$$

In typical structures, values of η may range near 10^{-3} to 10^{-2}. Even when great effort has been made to increase damping by special treatments, it is an extremely rare structure in which η is as large as 0.1. Other measures of damping with which the reader may be more familiar are directly related to the loss factor. The mechanical engineer's "damping ratio" (ratio of damping coefficient to a "critical" value) is

$$c/c_c = \frac{1}{2}\eta \quad ;$$

the electrical engineer's "Q" is

$$Q = 1/\eta \quad .$$

The exact solution of Eq. II.2.1 for damped natural vibration is

$$v(t) = V_o\, e^{-\frac{1}{2}\eta\omega_o t} \cos(\omega_o' t + \phi)$$

$$x(t) = (V_o/\omega_o)\, e^{-\frac{1}{2}\eta\omega_o t} \sin(\omega_o' t + \phi - \sin^{-1} \tfrac{1}{2}\eta) \qquad \text{(II.4.4)}$$

$$\omega_o' = \omega_o(1 - \tfrac{1}{4}\eta^2)^{1/2} \approx \omega_o$$

where V_o and ϕ are arbitrary constants determined from initial conditions. The equations describe sinusoidal vibrations with an amplitude that decreases slowly and exponentially in time. The frequency ω_o' differs from the undamped natural frequency ω_o by an amount which is insignificant when η is small.

Because of the damping, none of the energy functions is constant -- not even E; they all exhibit exponential decay and some short-time fluctuations. Short-time averaging, over one half-period or any integral number thereof, eliminates the short-time fluctuations but leaves the exponential decay. Indeed, the short-time average energies are found to be related in the same way as for the undamped resonator; only the gradual decay must be added to complete the picture:

$$\langle v^2 \rangle = \tfrac{1}{2} V_o^2\, e^{-\eta\omega_o t}$$

$$\langle x^2 \rangle = \langle v^2 \rangle / \omega_o^2$$

$$\langle T \rangle = \tfrac{1}{2} M\langle v^2 \rangle = \tfrac{1}{2} K\langle x^2 \rangle = \langle U \rangle \qquad \text{(II.4.5)}$$

$$\langle L \rangle = 0$$

$$\langle E \rangle = \langle T \rangle + \langle U \rangle = 2\langle T \rangle = M\langle v^2 \rangle \quad .$$

(These expressions are approximations valid for small damping, i.e. $1/4\eta^2 \ll 1$.) Finally, the short-time average power dissipated (Eq. II.2.3) is found to be

$$\langle -dE/dt \rangle = \langle \Pi \rangle = R\langle v^2 \rangle = \eta\omega_o M\langle v^2 \rangle = \eta\omega_o \langle E \rangle \quad ,$$

so that the decay of energy (strictly, its short-time average value) is described by

$$\langle E \rangle = E_o \, e^{-\eta\omega_o t} \qquad \text{(II.4.6)}$$

where E_o is the energy at time $t=0$. The solution for loss factor,

$$\eta = \langle \Pi \rangle / \omega_o \langle E \rangle \quad , \qquad \text{(II.4.7)}$$

relates η to the fraction of total energy that is dissipated is one cycle ($2\pi/\omega_o$ seconds) of the oscillation.

The coefficient of time in Eq. II.4.6 for the decay of energy is the <u>decay rate</u>

$$\alpha' \equiv \eta \; \omega_o \text{ nepers/sec} \quad , \qquad \text{(II.4.8a)}$$

one neper being the dimensionless ratio equal to $1/e$. However, in experimental work, it is customary to use powers of 10, rather than of e, to express exponentially decaying functions. Corresponding to Eq. II.4.6 one writes

$$E = E_o (10^{0.1})^{-\alpha t} \quad ,$$

whence it follows that

$$\alpha = 4.343 \; \eta \; \omega_o \text{ dB/sec} \quad . \qquad \text{(II.4.8b)}$$

The decibel (dB) is a dimensionless ratio of power-like quantities equal to $1/10^{0.1} \approx 1/1.26$.

Another measure of damping, commonly used in acoustics, is the <u>reverberation time</u>, defined as the time T_R required for the energy level to decrease by 60 dB (i.e. an energy ratio of $1/10^6$). It follows from Eq. II.4.8b that

$$\begin{aligned} \alpha T_R &= 60 \text{ dB} \\ T_R &\approx 2.2/\eta f_o \quad \text{sec} \end{aligned} \qquad \text{(II.4.8c)}$$

where $f_o = \omega_o/2\pi$ is the natural frequency in cps. In experimental work, the reverberation time is commonly inferred from the initial decay rate on the assumption that the decay is exponential; measurements over the whole 60 dB range are seldom made. This lax usage of the precise term is a firmly established custom.

II.5 Pure-Tone Variables; Complex Convention

Some prosaic matters of notation must precede consideration of the forced motion of a simple resonator. Response to steady excitation by a simple harmonic ("pure tone") force is an important special case of the general response problem. Pure tones are idealizations of reality which are invaluable in two ways. Many laboratory experiments and some practical problems involve almost-pure tones. Secondly, the solutions for pure tones are applicable to the spectral components of forces that have more complicated dependence on time.

Throughout this text we shall use the standard complex convention to denote simple harmonic variables. Suppose a force varies harmonically in time with a frequency ω rad/sec. Then, with complete generality, the force can be written

$$f(t) = A\cos(\omega t + \phi) \quad , \qquad \text{(II.5.1)}$$

where the amplitude A and phase angle ϕ are constant. In complex notation, one writes the force as a complex quantity,

$$f'(t) = F\, e^{i\omega t} \quad , \qquad \text{(II.5.2)}$$

where F is a constant complex number; but, one <u>means</u> by this notation that the physical force (a real number) is the real part of this complex quantity:

$$f(t) = \mathrm{Re}\left\{f'(t)\right\} = \mathrm{Re}\left\{F\, e^{i\omega t}\right\} \quad . \qquad \text{(II.5.3)}$$

The formula reduces to Eq. II.5.1 if one equates

$$F = A\ e^{i\phi} \quad . \tag{II.5.4}$$

Moreover, if the real part of the complex variable is a solution to the pure-tone problem, the imaginary part is also a solution. This result is shown by the following calculations:

$$\begin{aligned} \mathrm{Im}\left\{A\ e^{i\phi}\ e^{i\omega t}\right\} &= A \sin(\omega t + \phi) = \\ & A \cos(\omega t + \phi - \tfrac{1}{2}\pi) = \\ & A \cos[\omega(t - \tfrac{1}{4}\tau) + \phi], \text{where } \tau = 2\pi/\omega \quad . \end{aligned}$$

Thus, the imaginary part differs from the real part only in that the origin for measuring time has been shifted by a quarter of the period of oscillations, a difference of no significance. Of course, one must consistently use either the real or the imaginary parts of all variables in a single problem. The usual convention takes the real part.

The complex convention greatly simplifies calculations since one letter, F, stands for two, A and ϕ, and since exponential factors are so easily differentiated and integrated. Because the operations of taking the real and imaginary parts commute with linear differential and integral operations, e.g.

$$\frac{d}{dt}\ \mathrm{Re}\left\{f'(t)\right\} = \mathrm{Re}\left\{\frac{d}{dt}\ f'(t)\right\} \quad ,$$

the differential equation satisfied by the complex variable is formally identical with the equation satisfied by the real, physical variable.

There is one situation in which the complex notation of pure-tone variables can lead the innocent to grief -- when the product of two real, physical variables is required. One must beware of simply multiplying the complex numbers!

Products of complex variables can be used, however, to obtain the time-average of the product of the corresponding pure-tone real variables. An example of such a product is the kinetic energy, $1/2\ Mv^2$, but more general examples involve the product of two different variables. Consider two real, pure-tone variables

$$f(t) = F\cos(\omega t + \phi)$$

$$v(t) = V\cos(\omega t + \alpha)$$

which may be the applied force and the velocity of response. The product f v is the instantaneous power; it equals a constant part, the time-average power, and a time-varying part which is simply harmonic with the second harmonic frequency 2ω:

$$\Pi = fv = \frac{1}{2} FV\cos(\phi - \alpha) + \frac{1}{2} FV\cos(2\omega t + \phi + \alpha)$$

$$\langle\Pi\rangle = \langle fv\rangle = \frac{1}{2} FV\cos(\phi - \alpha) \quad .$$

In complex notation, the force and velocity would be written

$$f'(t) = F\, e^{i\phi}\, e^{i\omega t}$$

$$v'(t) = V\, e^{i\alpha}\, e^{i\omega t}$$

it being understood that the actual physical variables, f and v, are the real parts of these. The general mathematical theorem for the average product of the real pure-tone variables is

$$\langle fv\rangle = \langle \mathrm{Re}\left\{f'\right\} \mathrm{Re}\left\{v'\right\}\rangle = \frac{1}{2}\mathrm{Re}\left\{f'v'^*\right\} = \frac{1}{2}\mathrm{Re}\left\{f'^*v'\right\} \qquad \text{(II.5.5)}$$

where the superscript star is used to indicate the complex conjugate of the variable to which it is appended. The theorem is readily demonstrated from the relations outlined above.

II.6 Forced Sinusoidal Motion; Admittance; Resonances

We now consider the response of the simple damped resonator to a steady pure-tone force of arbitrary frequency. In complex notation, the applied force and the velocity of response are denoted

$$f'(t) = F\, e^{i\omega t} \qquad \text{(II.6.1)}$$

$$v'(t) = V\, e^{i\omega t} \quad .$$

We must find V in terms of F. In general V will be complex even if F is real. The relationship is found by substituting Eqs. II.6.1 into the differential equation, II.2.1, a process that eventually yields

$$F = V\ Z \tag{II.6.2}$$

where

$$Z = R + i(\omega M - K/\omega) \quad . \tag{II.6.3}$$

The exponential function of time is a common factor and cancels out.

The complex function Z, whose value depends on the driving frequency ω, is called the impedance of the simple resonator. It contains all the information required to compute response to a pure-tone force, via Eq. II.6.2. More often, we shall desire values for the reciprocal of Z, called the admittance

$$\begin{aligned} Y &= 1/Z \\ V &= F\ Y \quad . \end{aligned} \tag{II.6.4}$$

Both impedance and admittance can be given interpretations in terms of the energy functions defined in section 2. From the rule for time averaging with complex notation, Eq. II.5.5, we find the mean-square values of the real physical velocity and force:

$$\langle v^2 \rangle = \langle [\mathrm{Re}\{v'(t)\}]^2 \rangle = \frac{1}{2}\,\mathrm{Re}\{VV^*\} = \frac{1}{2}VV^* = \frac{1}{2}|V|^2$$

$$\langle f^2 \rangle = \langle [\mathrm{Re}\{f'(t)\}]^2 \rangle = \frac{1}{2}|F|^2 \quad .$$

Consider the quantity $Z\langle v^2\rangle$. When it is expanded by means of Eq. II.6.3, the individual terms can be identified with time-averages of the energy functions Π, T, and U (Eqs. II.2.2, II.2.3). The result is

$$J \equiv Z\langle v^2\rangle = \langle \Pi \rangle + i2\omega\langle L\rangle \quad . \tag{II.6.5}$$

The quantity J defined by Eq. II.6.5 is sometimes called the complex power delivered by the force to the resonator. Its real part is seen to be the time-averaged dissipated power, sometimes called the real power:

$$\langle \Pi \rangle = R\langle v^2\rangle \quad .$$

The imaginary part, called the reactive power, is proportional to the time-averaged Lagrangian

$$\langle L\rangle = \langle T-U\rangle = \tfrac{1}{2}M\langle v^2\rangle - \tfrac{1}{2}K\langle x^2\rangle =$$

$$\tfrac{1}{2}M\langle v^2\rangle - \tfrac{1}{2}K\langle v^2\rangle/\omega^2 =$$

$$\tfrac{1}{2}M\langle v^2\rangle\,(1-\omega_o^2/\omega^2) \quad .$$

When the driving frequency ω equals the natural frequency of the undamped resonator, ω_o, then $\langle L\rangle = 0$ and the complex power J, the impedance Z, and the admittance Y are all real. (Of course, this is more readily proved directly from Eq. II.6.3.) The equality of time-averaged kinetic and potential energies which is implicit in the vanishing of $\langle L\rangle$ has been seen earlier to be a feature of natural, unforced vibration of the simple resonator, both with damping and without. This condition is defined as resonance, and ω_o is called the resonance frequency.

Since the average values of T and U are equal at resonance, the time-average of total energy is

$$\langle E\rangle = 2\langle T\rangle \quad .$$

It follows that the loss factor η has the same energetic interpretation for steady response at resonance as we found in Eq. II.4.7 for natural damped vibrations,

$$\eta = R/\omega_o M = \langle \Pi\rangle/\omega_o\langle E\rangle \ ;$$

i.e., η is proportional to the fraction of total energy lost in one cycle.

The complex power J can be shown by Eqs. II.6.2 and II.6.4 to be also equal to

$$J = Z\langle v^2\rangle = \tfrac{1}{2}Z|V|^2 = \tfrac{1}{2}FV^* = \tfrac{1}{2}Y^*|F|^2 = Y^*\langle f^2\rangle \quad . \qquad \text{(II.6.6)}$$

Thus the impedance Z and admittance Y are related to values of the energy functions for unit values of rms velocity and rms force respectively:

$$Z = [\langle \Pi \rangle + i2\omega \langle L \rangle]_{\langle v^2 \rangle = 1}$$
$$Y = [\langle \Pi \rangle - i2\omega \langle L \rangle]_{\langle f^2 \rangle = 1} \quad . \tag{II.6.7}$$

These relationships between impedance and admittance functions and the response energy functions have been demonstrated here for a single simple resonator. However, they are also valid for the response of a general linear system with viscous dissipation when the excitation is describable by a single pure-tone variable. (The restriction requires either that the system be driven at a single point or that $f(t)$ be a generalized force and $v(t)$ a generalized velocity.) Indeed, the proofs offered in texts oriented to electronics all start with a general system.*

In many situations, the time-averaged power loss $\langle \Pi \rangle = \mathrm{Re}(J)$ is the most important factor in an analysis. Equations for this real part corresponding to Eq. II.6.6 are

$$\langle \Pi \rangle = \langle v^2 \rangle \, \mathrm{Re}(Z) = \langle v^2 \rangle \, R = \frac{1}{2} \mathrm{Re}(FV^*) = \langle f^2 \rangle \, \mathrm{Re}(Y) \quad . \tag{II.6.8}$$

The real part of admittance has a special name, the <u>conductance</u>, and a symbol G, so that;

$$\langle \Pi \rangle = \langle f^2 \rangle \, G \quad . \tag{II.6.9}$$

The particular importance of the real power Π resides in its significance as the rate at which energy passes out of the system, having been first fed into it through the action of the applied force. As pointed out in the discussion following the introduction of the symbol Π in Eq. II.2.3, the manner in which this energy is "lost" is of no consequence to the analysis of the simple resonator. The energy may be lost by conversion to heat, or it may pass on to other systems where it appears as vibrational energy, kinetic or potential. This latter example, in which the simple resonator is a transmission system between the source and other structures, is the case of greater practical interest.

*For example, see E. A. Guillemin, <u>Theory of Linear Physical Systems</u> (John Wiley and Sons, New York, 1963), Chapters 5 and 8.

However, we cannot always completely ignore the influence of the reactive forces, that is, the imaginary parts of J and Y. In particular, determination of the response amplitude of the resonator under the action of specified forces requires knowledge of the magnitude of the admittance Y; for (cf. Eq. II.6.4)

$$\langle v^2 \rangle = |Y|^2 \langle f^2 \rangle \quad . \tag{II.6.10}$$

A plot of $|Y(\omega)|$ as a function of frequency is called the <u>response curve</u>.

The response curve of a simple resonator with small damping is dominated by a single large, narrow peak at the resonance frequency, from which the curve falls off smoothly in both directions (Fig. II.2). The curves for all resonators are very much alike when the admittance is written in normalized form using loss factor and a normalized frequency ratio $\bar{\omega}$:

$$\omega_o MY = [Z/\omega_o M]^{-1} = [\eta + i(\bar{\omega} - 1/\bar{\omega})]^{-1} \tag{II.6.11}$$

where $\quad \eta \equiv R/\omega_o M$ and $\bar{\omega} \equiv \omega/\omega_o \quad .$

The whole range of frequencies is conveniently divided into three parts. In the <u>resistance-controlled</u> region near resonance, the admittance is mainly determined by the resistance, i.e. the dashpot. The spring forces are most important at lower driving frequencies, in the <u>stiffness-controlled</u> region, and inertial forces predominate in the <u>mass-controlled</u> region at higher frequencies. The boundaries between these regions are the frequencies at which the real and imaginary parts of Y (or of Z) are equal in magnitude, that is, the roots of

$$\eta^2 = (\bar{\omega} - 1/\bar{\omega})^2$$

which are

$$\bar{\omega} \approx 1 \pm \tfrac{1}{2}\eta \tag{II.6.12}$$

in the case of small damping.

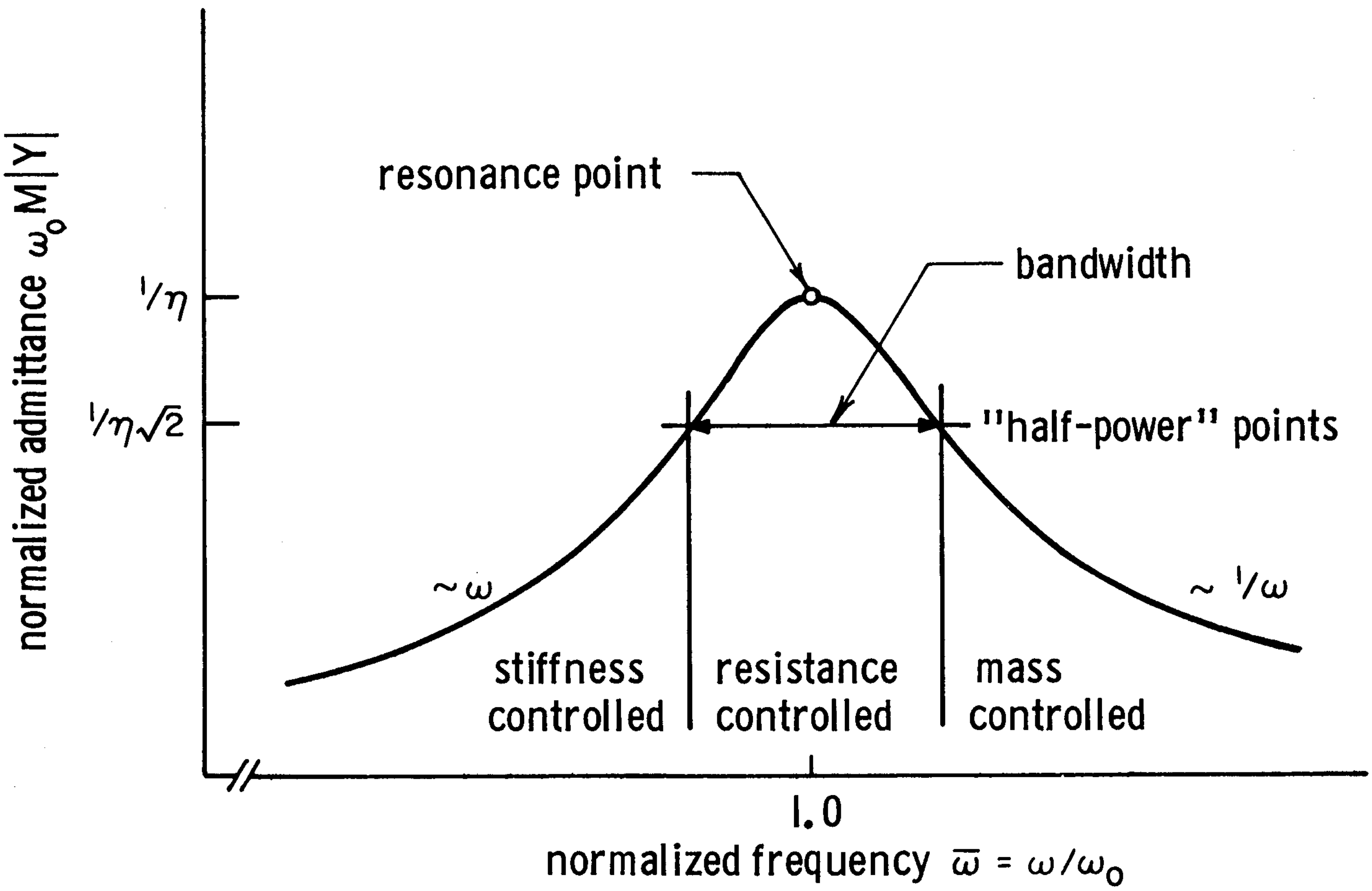

Figure II.2.- Resonance curve of simple resonator.

These boundary frequencies have further significance in relation to the magnitude of the admittance. The maximum admittance occurs at the resonance frequency:

$$|Y|_{max} = 1/\omega_o M\eta = 1/R = G_{max} \quad . \qquad \text{(II.6.13)}$$

Since the admittance is real at resonance, it then equals the conductance. Note that the response velocity of a resonator at resonance bears a simple relation to the response that the mass alone would have if driven by the same force. In the latter case,

$$v_{rms} = f_{rms}/\omega_o M \quad ,$$

a simple restatement of Newton's law: f=Ma. For the resonator driven at resonance, the response

$$v_{rms} = f_{rms}/\omega_o M\eta$$

is larger by the factor η^{-1}. For this reason, η^{-1} is called the <u>resonant amplification</u>.

Now, at the frequencies bounding the resistance-controlled region, the magnitude of the admittance is smaller than the maximum value by a factor $1/\sqrt{2}$:

$$|Y| = 1/|Z| = |\omega_o M[\eta + i\eta]|^{-1} =$$

$$[\omega_o M\eta\sqrt{2}]^{-1} = 1/\sqrt{2}R \quad .$$

On the other hand, the conductance at each boundary frequency is smaller than its maximum value by a factor 1/2:

$$G = \mathrm{Re}(Y) = \mathrm{Re}(1/Z) = \mathrm{Re}(Z)/|Z|^2 = R|Y|^2 = 1/2R \quad . \qquad \text{(II.6.14)}$$

This latter feature yields a name for the two frequencies: the <u>half-power points</u> of the resonance curve. Suppose that the resonator is driven by a force of constant amplitude but variable frequency. The maximum amplitude of response obtains at resonance where $|Y|$ is a maximum (see Eq. II.6.10); the power $\langle\Pi\rangle$ is a maximum at the same frequency (Eq. II.6.9). At the frequencies of the half-power points, the power dissipated is half this maximum value.

The frequency interval between the half-power points is a convenient measure of the range of frequencies in which the resonator is "sensitive" to excitation, and responds freely. For small damping, this frequency interval, the <u>half-power bandwidth</u>, is

$$\Delta_{1/2}(\omega) = \eta\omega_o = R/M \text{ rad/sec} \quad . \tag{II.6.15}$$

(See Eq. II.6.12.) Note that this bandwidth is numerically equal to the decay rate for energy of the resonator in natural vibration (Eq. II.4.8). We see that the various measures of damping and "strength of resonance" are all closely interrelated.

<u>Numerical Example</u>: For example, consider a simple resonator which is resonant at 100 cps (200π rad/sec) with a loss factor $\eta=10^{-2}$ -- a typical average value for structures with no special damping treatment. We have, then, a half-power bandwidth of

$$\Delta_{1/2} = \eta\omega_o = 2\pi \quad \text{rad/sec}$$

or 1 cps. The principal response to a force of adjustable frequency lies in the resistance-controlled region of the response curve between 99.5 cps and 100.5 cps. Now, if the structure is set into oscillation and left alone, the energy of natural vibrations decays exponentially at a rate (Eq. II.4.8).

$$\alpha' = \eta\omega_o = 2\pi \text{ neper/sec}$$

$$\alpha = 4.343\alpha' = 27.3 \text{ dB/sec} \quad .$$

The energy drops to half its initial value in about 1/9 sec, or 11 cycles; after 1 sec, less than 0.002 of the energy remains; the reverberation time is 2.2 sec. These numbers may reasonably suggest a "speedy" decay; however, as structures come, this one is only moderately resonant. Note the narrow band of frequencies to which the principal response is limited; this particular structure knows its pitch about as well as a mediocre baritone.

II.7 <u>Random Excitation; Spectral Analysis</u>

The vibratory forces acting on structures around us are seldom steady pure tones. Indeed, they never are, in a strict sense, since there is always a starting time or an ending, or some intermediate fluctuation that affects the purity of the force. But in a more practical sense, many physical mechanisms are exceedingly erratic and complex, exhibiting none of the regularity of pure tones. The noise of rockets, jet engines, and turbulent flow are examples. In such cases, reasonable statistical predictions of response can be made if the characteristics of the force are sufficiently steady during any one experiment (or between repetitions of an experiment) to allow a statistical description of it to be formed.

Even when the precise characteristics of a variable are known, it may be convenient to pretend ignorance of some factors. For example, at any instant t, there is a unique value of the velocity of natural damped vibration (Eq. II.4.4)

$$v(t) = V_o \, e^{-\frac{1}{2}\alpha t} \cos(\omega t + \phi) \quad .$$

However, the short-time mean square value (Eq. II.4.5),

$$\langle v^2 \rangle = \frac{1}{2} V_o^2 \, e^{-\alpha t} \quad ,$$

is the statistical average (<u>ensemble average</u>, <u>expected value</u>, or <u>expectation</u>) of $v^2(t)$ for the ensemble of repeated experiments in which ϕ assumes different values, every value being equally probable. From another viewpoint, the rms value $\langle v^2 \rangle^{1/2}$ is the "best" estimate for the magnitude of v(t) in a single experiment, if the value of ϕ is unknown. The statistical average eliminates details of the response which are dependent upon an unpredictable condition, the value of ϕ.

Very little statistical sophistication is required for the present study. Our dominant interest is the ability to predict the rms response to steady forces. Because of the resonant nature of the structures, it is essential that we be able to describe the "distribution in frequency" (i.e. the frequency spectrum) of both force and response. The present summary of the statistical analysis of complex signals is restricted to these points of interest, and the reader is referred elsewhere for analytical details and for other aspects of the general topic.*

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*For example, see W. B. Davenport and W. L. Root, <u>Random Signals and Noise</u> (McGraw-Hill Book Company, New York, 1958) or the briefer treatment in S. H. Crandall, <u>Random Vibrations</u>, Vol. 1 (The Massachusetts Institute of Technology Press, Cambridge, 1959), especially Chapter 4.

Consider a sample of finite duration of a particular force, however complex or simple. This forcing function can be expressed as a superposition of steady pure-tone components, of infinite duration, by the analytical procedures of the Fourier transform. The components are continuously distributed in frequency; no finite number of discrete components is adequate to describe the force sample. The response of a structure to this force can be similarly described by a distribution in frequency. The relative strengths of the components of response and of exciting force at any one frequency are given by the admittance $Y(\omega)$ (Eq. II.6.3), since these components are steady pure tones.

The spectral description of either of these functions of time, the exciting force or the response, is definable operationally by the process of narrow-band analysis. Let the force be converted to an electrical signal $f(t)$ and directed to a narrow-band analyzer. The first part of the analyzer is a filter that transmits without attenuation any pure tone whose frequency lies in a band of small width Δ centered on the frequency ω; the filter attenuates completely any tone whose frequency does not lie in the band. The second part of the analyzer is a square-law averaging device whose output $E_f(\omega,\Delta,T)$ is the time-average (over the duration T of the sample) of the square of the filter's output.

For long, continuous functions of the type called stationary, the output $E_f(\omega,\Delta,T)$ approaches a well-defined limit as longer and longer samples are analyzed, i.e. at $T \rightarrow \infty$.* That limit

$$E_f(\omega,\Delta) \equiv \lim_{T\rightarrow\infty} E_f(\omega,\Delta,T)$$

is the long-time mean-square value of all the components of $f(t)$ which have frequencies in the bandwidth Δ centered on ω. The spectral density of $f(t)$ is the quantity obtained by first dividing $E_f(\omega,\Delta)$ by the bandwidth, and then taking the limit of ever smaller bandwidths:

$$S_f(\omega) = \lim_{\Delta\rightarrow 0} E_f(\omega,\Delta)/\Delta \quad .$$

*Of course, no real signal can be truly stationary, if only because time is limited. The consequent difficulties and uncertainties in analysis and experiment have much practical importance, especially in the case of short times. However, only the ideal stationary signal is considered in this study.

The units of spectral density $S_f(\omega)$ are the units of f^2 divided by the units of bandwidth Δ. Thus, the spectral density of force has the units of $(\text{force})^2/(\text{rad/sec})$; for the spectral density of velocity, the units are those of $(\text{velocity})^2/(\text{rad/sec})$. Often, bandwidth is expressed in cycles per second (cps) instead of rad/sec; then S_f has the units of $(\text{force})^2/\text{cps}$.

Two theorems on the spectral analysis of stationary signals are central to the present study. The first relates the mean-square value of the total force to the spectral density:

$$\langle f^2(t)\rangle_t = \int_0^\infty S_f(\omega)\, d\omega \quad . \tag{II.7.1}$$

In terms of the operational description of spectral density, the theorem merely states that the mean-square value of the force is the sum of the mean-square values of all its components. Of course, the same relationship between mean-square value and spectral density holds for any other variable -- velocity, acceleration, etc.

The second central theorem relates the spectral densities of the excitation and of the response of any linear system. When a pure-tone force $f(t)$ is applied to any linear system the velocity of response $v(t)$ is a pure tone of the same frequency. The ratio of the complex amplitudes, V and F, defines the complex admittance function for the system:

$$Y(\omega) \equiv V/F \quad . \tag{II.7.2a}$$

In the same way as for the simple resonator studied in the previous section, the square of the magnitude of Y relates the mean-square values of the pure-tone $v(t)$ and $f(t)$:

$$\langle v^2\rangle = |Y(\omega)|^2 \langle f^2\rangle \quad .$$

Now, in the case of a complicated force function such as random noise, where there are no components with discrete frequencies, this same relationship holds for the spectral "components". Thus, if $S_f(\omega)$ is the spectral density of any stationary force, the spectral density of the response of the system is given by

$$S_v(\omega) = |Y(\omega)|^2\, S_f(\omega) \quad . \tag{II.7.2b}$$

This second fundamental theorem expresses the equality between the ratio of spectral densities in random response and the squared-magnitude of the pure-tone response function. On reflection, this is seen to be a general theorem for any two related variables. Let $v(t)$ and $f(t)$ be any two variables associated with the response of a system. In pure-tone response, they are related by an equation like Eq. II.7.2a. Then, in stationary random response, their spectral densities are related by Eq. II.7.2b.

For example, in many structural problems, one desires to characterize the response by the displacement or the acceleration, instead of by the velocity. At every instant, the acceleration is the time derivative of velocity, which is itself the time derivative of amplitude. In the complex notation for pure tones, a single time derivative is equivalent to multiplication by $(i\omega)$. It follows directly from the second theorem that the spectral densities of the three response quantities are, in all cases, very simply and exactly related by

$$S_a(\omega) = \omega^2 S_v(\omega) = \omega^4 S_x(\omega) \quad , \qquad \text{(II.7.3)}$$

where the subscripts a and x stand for acceleration and displacement.

In many circumstances, the two theorems are combined. Let $v(t)$ be the response velocity to a force $f(t)$. In the case of pure tones, they are related by the admittance $Y(\omega)$. In the case of stationary random noise, the spectral densities are related by Eq. II.7.2b. Then the mean-square response velocity can be computed from its spectral density by a formula of the type of Eq. II.7.1. The result is

$$\langle v^2(t)\rangle_t = \int_0^\infty |Y(\omega)|^2 \, S_f(\omega) \, d\omega \quad . \qquad \text{(II.7.4)}$$

We proceed to apply these general relations to the simple resonator, investigating a variety of different excitations.

II.7.a. Broad-Spectrum Force; Effective Bandwidth

Random forces characteristically have spectral densities which are continuous functions of frequency. The simplest example is "white noise," whose spectral density is constant for all frequencies. White noise is an important idealization never met in practice. However, the spectra of actual forces are often such broad, reasonably flat curves that they might as well be considered flat for the purpose of computing response. Such is often the case for rocket and jet noises. Let us examine qualitatively the approximation involved in replacing a broad spectrum by an equivalent flat spectrum.

If the force spectrum is truly flat, the constant value of S_f can be taken outside the integral in Eq. II.7.4. This simplifying procedure is also justifiable, as a good approximation, if S_f is constant over all frequencies where the admittance is large. The admittance of a simple resonator has been found to be large only in a small region of frequency, a bandwidth equal typically to a few cycles per second (Eq. II.6.15). Moreover, $|Y|^2$ falls off fast and smoothly on both sides of the peak, as shown in Fig. II.3. Therefore, unless S_f has a <u>very</u> large peak at some other frequency, the integral (II.7.4) is dominated by the contributions at frequencies near resonance. Finally, we note that $|Y|$ is very closely symmetrical about the resonance frequency ω_o, when the loss factor η is small, so that even the slope of the spectrum curve will have no significant effect on the integral. Thus we justify, in a qualitative manner, replacing a broad spectrum $S_f(\omega)$ in Eq. II.7.4 by a constant equal to its value $S_f(\omega_o)$ at the resonance frequency.

Now, we proceed to calculations. The mean-square response of the simple resonator to a broad-spectrum random force is, then, very closely equal to

$$\langle v^2 \rangle = S_f(\omega_o) \int_0^\infty |Y(\omega)|^2 \, d\omega \quad ,$$

where Y is given in Eq. II.6.11. When it is noted that $|Y(\omega)| = |Y(-\omega)|$ is an even function of ω, the integral can be readily performed in the complex plane (by application of Cauchy's residue theorem) with the result:

$$\int_0^\infty |Y|^2 \, d\omega = \pi/2RM \quad . \qquad \text{(II.7.5)}$$

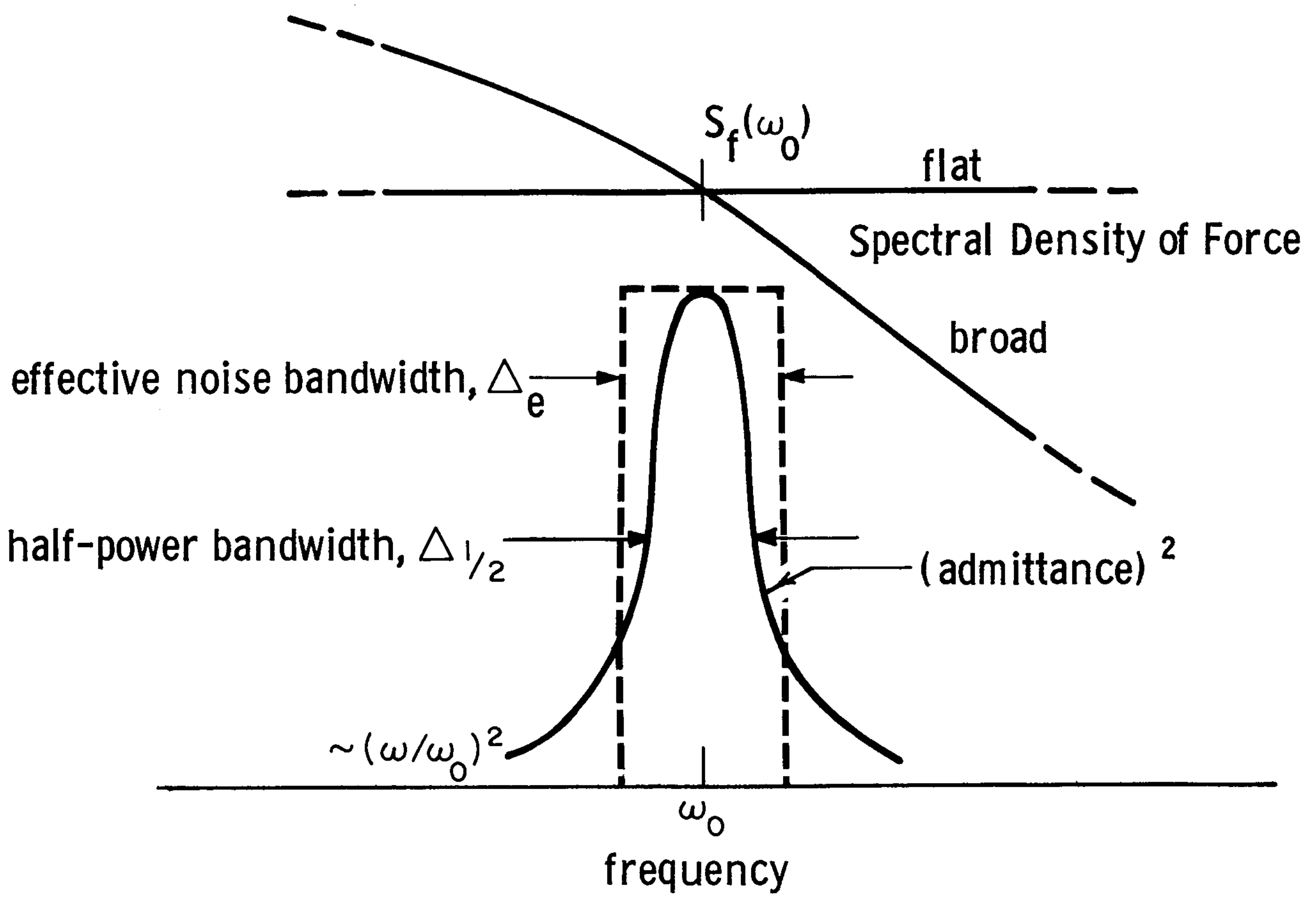

Figure II.3.- Spectral density of random force and square of admittance of simple resonator.

Thus, the mean-square velocity is given by

$$\langle v^2 \rangle = S_f(\omega_o)\pi/2RM \quad . \tag{II.7.6}$$

This simple formula will be useful throughout this study.

The mean-square values of displacement and acceleration can be computed from their spectral densities (Eq. II.7.3) by similar integrals. The results are simply

$$\begin{aligned} \langle x^2 \rangle &= \langle v^2 \rangle / \omega_o^2 \quad , \\ \langle a^2 \rangle &\approx \omega_o^2 \langle v^2 \rangle \quad , \end{aligned} \tag{II.7.7}$$

relations which are identical with those governing response to a pure tone at the resonance frequency ω_o. The correspondence is not entirely coincidental, for the response to noise of a lightly damped resonator resembles a pure tone with nearly the resonance frequency and with an amplitude that fluctuates slowly. The first relation in Eq. II.7.7 is not restricted to small damping. The second relation is an approximation for small damping in which the main error is the neglect of a non-resonant acceleration response. This additional acceleration, whose mean-square value is

$$\langle a'^2 \rangle = \langle f^2 \rangle / M^2 \quad ,$$

equals the response of the mass alone -- no spring or dashpot -- to the total, broad-band force. It is usually negligible. The total mean-square acceleration is the sum of the two parts.

The correspondence between response to noise and response to a pure tone at the resonance frequency is sometimes made even more explicit by introducing an <u>effective rms force</u> f_e defined by the pure-tone resonance relation:

$$f_e^2 \equiv \langle v^2 \rangle R^2 \tag{II.7.8}$$

The value of the effective force is found from Eq. II.7.6. It is given by

$$f_e^2 = S_f(\omega_o)\Delta_e \tag{II.7.9}$$

where Δ_e is an <u>effective bandwidth</u>. The value of the effective bandwidth of the simple resonator is found to be

$$\Delta_e = \tfrac{1}{2}\pi\omega_o\eta = \tfrac{1}{2}\pi R/M = \tfrac{1}{2}\pi\Delta_{1/2} \text{ rad/sec} \quad . \tag{II.7.10}$$

(Here, $\Delta_{1/2}$ is the half-power bandwidth, Eq. II.6.15.) Equation II.7.9 indicates that the effective force equals the rms value of those components of the actual force that lie in a band Δ_e centered on the resonance frequency. Such a band is sketched in Fig. II.3.

Finally, consider the energetics of the simple resonator driven by a broad-spectrum force. From simple time averages of the instantaneous relations in Eqs. II.2.2 and II.2.3, we can find the time averages of kinetic energy T, potential energy U, and power Π:

$$\begin{aligned} \langle T\rangle &= \tfrac{1}{2}M\langle v^2\rangle \\ \langle U\rangle &= \tfrac{1}{2}K\langle x^2\rangle = \tfrac{1}{2}M\omega_o^2\langle x^2\rangle = \langle T\rangle \\ \langle \Pi \rangle &= R\langle v^2\rangle \end{aligned} \tag{II.7.11a}$$

The equality of kinetic and potential energies follows from Eq. II.7.7. Expressions for total energy E, Lagrangian L, and loss factor η follow directly:

$$\begin{aligned} \langle E\rangle &= \langle T\rangle + \langle U\rangle = 2\langle T\rangle = M\langle v^2\rangle \\ \langle L\rangle &= \langle T\rangle - \langle U\rangle = 0 \\ \eta &= R/\omega_o M = \langle \Pi\rangle/\omega_o\langle E\rangle \quad . \end{aligned} \tag{II.7.11b}$$

All of these relations are identical to those found for response to a pure-tone force at the resonance frequency.

<u>Numerical Example</u>: As an example of the relations developed in this section, consider a simple resonator which is resonant at 100 cps with a loss factor $\eta=10^{-2}$, and with a mass of 1.0 kg. Suppose it to be excited by a broad-spectrum random noise. The applied force is measured; a calibrated transducer creates an electrical signal proportional to force, which is filtered and metered. An rms force of 10^5 dynes (0.225 lb) is measured for a filter bandwidth of 10 cps centered on 100 cps.

This measurement includes all components of S_f in the range from 95 to 105 cps. On the assumption that the force spectrum is flat in this region, one computes

$$10^{10}\ \text{dynes}^2 = S_f(\omega_o)\int_{180\pi}^{210\pi} d\omega$$

$$S_f \approx 10^9/2\pi\ \text{dyne}^2/(\text{rad/sec}) \quad .$$

The effective bandwidth (Eq. II.7.10) is

$$\Delta_e = \frac{1}{2}\pi\omega_o\eta = \pi^2\ \text{rad/sec} \quad .$$

Thus, the rms effective force (Eq. II.7.9) is

$$f_e = \left[\frac{1}{2}\pi \times 10^9\right]^{1/2}\ \text{dynes}$$

$$\approx 4 \times 10^4\ \text{dynes}$$

or 0.09 lb. The resistance R is to be found from the loss factor and mass:

$$R = \eta\omega_o M = 2\pi \times 10^3\ \text{dyne sec/cm}$$

in cgs units. The rms velocity of response is, then,

$$\langle v^2\rangle^{1/2} = f_e/R \approx 6.3\ \text{cm/sec}$$

and the displacement and acceleration are

$$\langle x^2\rangle^{1/2} = \langle v^2\rangle^{1/2}/\omega_o \approx 10^{-2}\ \text{cm}$$

$$\langle a^2\rangle^{1/2} \approx \omega_o\langle v^2\rangle^{1/2} \approx 4 \times 10^3\ \text{cm/sec}^2$$

or 4 "g", i.e. 4 times the standard acceleration of gravity. The power absorbed by the resonator is

$$\langle \Pi \rangle = R\langle v^2 \rangle = f_e^2/R = 2.5 \times 10^5 \text{ erg/sec}$$

or 1/40 watt.

The calculated acceleration does not include the non-resonant contribution which is proportional to the total rms force. (See Eq. II.7.7 and following discussion.) However, the usual unimportance of that contribution is readily shown by some numbers. The non-resonant contribution would equal the calculated resonant contribution only if the total rms force is

$$\langle f^2 \rangle^{1/2} = M\langle a^2 \rangle^{1/2} = 4 \times 10^6 \text{ dynes}$$

or 9 lb. For a force with constant spectral density, equal to the value calculated above, to have this large an rms value requires a bandwidth

$$B = \langle f^2 \rangle / S_f = 10^5 \text{ rad/sec}$$

or 16,000 cps. Seldom, if ever, does a mechanical force have constant spectral density to such high frequencies.

II.7.b Narrow-band Force; Dirac δ-Function

The farthest extreme from a force with flat or broad spectrum is one whose spectrum is extremely narrow. Consider a sequence of force spectra in which significant values are limited to ever narrower bands of frequency, centered on some frequency ω_1. In the sequence, let us hold constant the total rms force, i.e. the area under $S_f(\omega)$.

When the bandwidth of the force has been made much narrower than the bandwidth of the resonator to which it is applied, the force is effectively a pure-tone of frequency ω_1. The word "effectively" is a hedge, because the force retains some random character. It can be described as a sine wave whose frequency fluctuates slightly within the bandwidth centered on ω_1, and whose amplitude fluctuates slowly around the rms value. But the fluctuations are so slow that the response follows them quasi-steadily, i.e. the instantaneous amplitudes of force and response are governed by the pure-tone relations pertinent to the frequency ω_1 of the force.

The mathematical expression of this situation is the following. When the bandwidth of the force spectrum is small compared with the bandwidth of admittance, the admittance can be considered constant in the integral (Eq. II.7.3) for response:

$$\langle v^2\rangle = \int |Y(\omega)|^2 \, S_f(\omega)d\omega \approx |Y(\omega_1)|^2 \int S_f(\omega)d\omega =$$

$$|Y(\omega_1)|^2 \langle f^2\rangle \quad ,$$

which is the pure-tone relation (Eq. II.6.10).

The mathematical notation of the spectral density, in the limit of the sequence as the bandwidth approaches zero, involves the Dirac delta function;* one writes

$$S_f(\omega) = \langle f^2\rangle \, \delta(\omega-\omega_1) \quad . \tag{II.7.12}$$

The delta function $\delta(x)$ is zero everywhere except at $x=0$, where it is unlimited. However the integral of the product of a δ-function and any well-behaved "good" function G has the following value:

$$\int_{\omega_1-\epsilon}^{\omega_1+\epsilon} G(\omega) \; \delta(\omega-\omega_1)d\omega = G(\omega_1) \quad , \tag{II.7.13}$$

as long as the range of integration includes the frequency ω_1 on which the δ-function is centered. These definitions are readily shown to be equivalent to the calculation of $\langle v^2\rangle$ given above. The δ-function is a mathematical idealization of the narrow band spectrum; its physical significance is revealed by integration.

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*The delta function is discussed in more detail in most texts on Fourier or Laplace transforms; for example, see F. B. Hildebrand, Advanced Calculus for Applications (Prentice-Hall Inc., Englewood Cliffs, N. J., 1962), p. 63ff.

Incidentally, the δ-function notation of the force spectrum (Eq. II.7.12) is just as appropriate to a steady pure tone as to the limiting case of a random force with ever narrower bandwidth. In fact, the two excitations are indistinguishable. From a given sample of the signal, one cannot determine whether its amplitude will remain steady in the future, as for the pure tone, or will eventually fluctuate slowly about the rms value, as for the narrow-band random noise.

II.8 Multiple Forces

We have so far considered only the response of a single resonator to various single forces. The practical situation is seldom so simple. Very often, several forces are applied at the same time. In other circumstances, several forces are applied individually, at different times in different experiments, but one wants to find an average of the responses in the various experiments. In all cases, our interest focuses on the long-time average response, both the rms value and its spectral description.

In this section we consider a number of these more complicated situations involving averages with more than one force. The examples chosen are those with easy statistics. Fortunately, they include idealizations of the practical situations found to be most important.

II.8.a. Uncorrelated Forces

Consider a single simple resonator excited by two different forces, f_1 and f_2, so that the total force is their sum:

$$f(t) = f_1(t) + f_2(t) \quad . \qquad \text{(II.8.1)}$$

(Fig. II.4.) We wish to compare the response to the sum with the responses to the individual forces. First we consider the spectral densities of the forces themselves.

For general forces, the spectral density of the sum f will differ from the sum of the spectral densities of the parts, f_1 and f_2. We can write the spectral density of the sum in the form

$$S_f(\omega) = S_{f_1}(\omega) + S_{f_2}(\omega) + 2S_{12}(\omega) \qquad \text{(II.8.2)}$$

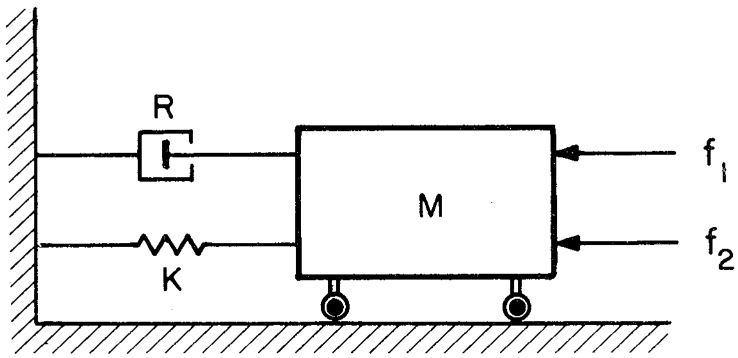

Figure II.4.- Several forces on a resonator.

where the "correction" term involves a function $S_{12}(\omega)$ which is called the cross-spectral density.*

Fortunately for simple calculations, there is a large class of forces for which the cross-spectral density vanishes. These are the uncorrelated forces, whose cross-correlation function, the time average

$$R_{12}(\tau) \equiv \langle f_1(t)\ f_2(t+\tau)\rangle_t \quad , \qquad \text{(II.8.3)}$$

vanishes for all values of the delay time τ. Two random forces generated by independent random mechanisms are uncorrelated, although their spectral densities can be identical. Furthermore, two forces whose spectral densities do not overlap are also uncorrelated. This follows directly from the concept that determining spectral density is equivalent to narrow-band filtering. In any frequency range where S_{f_1} vanishes, the output of a filter is independent of the presence of $f_1(t)$ at its input. Therefore, S_f must equal S_{f_2} at such frequencies, and S_{12} must vanish.

In summary, for two uncorrelated forces, the spectral density of their sum equals the sum of their individual densities. As a result, all the various mean-square properties of the response must be expressible as similar sums. For example, consider the mean square velocity in response to f:

$$\langle v^2\rangle = \int_0^\infty |Y(\omega)|^2\ S_f(\omega)\ d\omega \quad .$$

Evidently, this equals the sum of mean-square responses to the individual forces:

$$\langle v^2\rangle = \langle v_1^2\rangle + \langle v_2^2\rangle \quad .$$

*In a more general context, the cross-spectral density of any two variables is a complex function, of which the term defined by Eq. II.8.2 is the real part. The imaginary part is not required in the present context. Corresponding, the necessary conditions for $S_{12}=0$ is that only the even part of R_{12} (defined below) should vanish.

The same is true also for the energy, power dissipated, mean-square acceleration, etc.

These results for a pair of uncorrelated forces can be immediately extended to the simultaneous application of any number of uncorrelated forces. If the forces are uncorrelated, the total spectral density is the sum of the densities of individual components:

$$S_f(\omega) = \Sigma\, S_{f_n}(\omega) \quad . \tag{II.8.4a}$$

The spectral density of response is, therefore, also a sum of responses to individual components:

$$S_v(\omega) = \Sigma\, S_{v_n}(\omega) \quad . \tag{II.8.4b}$$

Similarly, all quantities related to mean-square response are given by sums:

$$\begin{aligned} \langle v^2\rangle &= \Sigma\langle v_n^2\rangle \\ \langle E\rangle &= \Sigma\langle E_n\rangle \\ \langle \Pi\rangle &= \Sigma\langle \Pi_n\rangle \quad . \end{aligned} \tag{II.8.4c}$$

In the special case of N uncorrelated forces with <u>identical</u> spectra, the sums can be replaced by N times the response to one force.

II.8.b Several Forces with Different Coupling

An important example of the previous results is afforded by the case where several forces are simultaneously applied to a resonator through different coupling mechanisms. A simple instance is illustrated in Fig. II.5, which shows the several forces applied to points on an ideal lever (rigid and massless) which is in contact with the mass of the resonator.

If the force applied at position z_1 on the lever is denoted $q_1(t)$, then the force on the mass due to this force on the lever is just

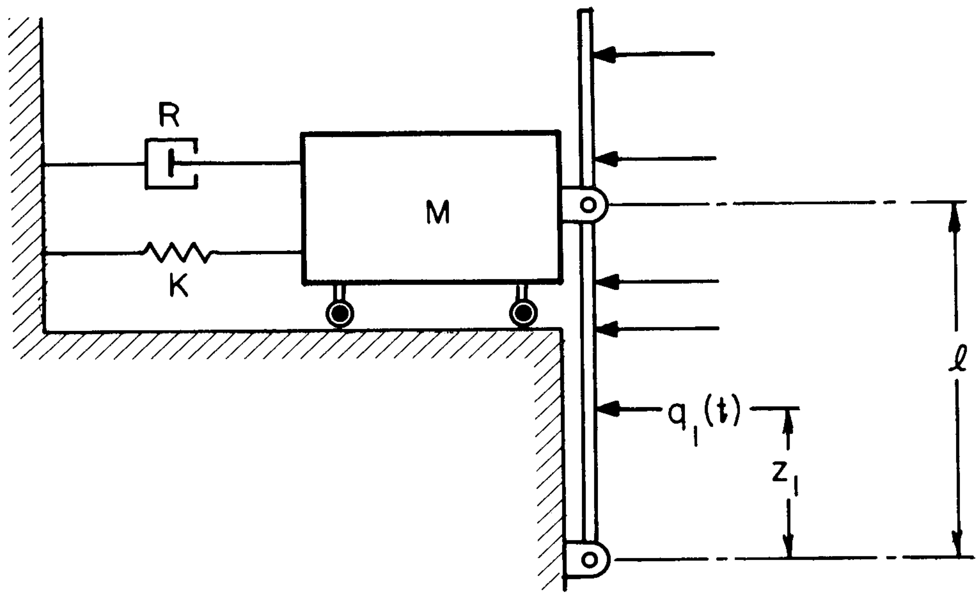

Figure II.5.- Several forces with different coupling (through a lever).

$$f_1(t) = H_1 q_1(t) \quad , \qquad H_1 \equiv z_1/\ell \quad , \tag{II.8.5}$$

where ℓ is the length of the lever. The constant coefficient H_1 will be called the <u>transfer function</u>. Let the spectral density of $q_1(t)$ be denoted by $S_{q_1}(\omega)$. Then, the spectral density of $f_1(t)$ is found directly by application of the second fundamental theorem, Eq. II.7.2:

$$S_{f_1}(\omega) = |H_1|^2 \, S_{q_1}(\omega) \quad . \tag{II.8.6}$$

In the present example, the transfer function is a real number, and the magnitude sign is superfluous.

<u>Uncorrelated forces</u>: Now consider the simultaneous application of several forces $q_n(t)$ at positions z_n. The total force on the mass is the sum of the contributions of each component

$$f(t) = \Sigma \, H_n \, q_n(t) \quad .$$

If the forces q_n are uncorrelated, and only then, the spectral density of the sum is the sum of spectral densities:

$$S_f(\omega) = \Sigma \, S_{f_n}(\omega) = \Sigma \, |H_n|^2 \, S_{q_n}(\omega) \quad . \tag{II.8.7}$$

The spectral density and mean-square value of response velocity can be found from S_f by application of the general theorems, Eqs. II.7.2 and II.7.4. The expressions will involve the admittance of the resonator, $Y(\omega)$.

<u>Uncorrelated forces with same spectrum</u>: In an interesting special case, all of the uncorrelated forces q_n have identical spectral densities. Such is the situation if each is derived from a nominally identical, but independent, random mechanism. Let N be the total number of forces and denote the sum of applied forces by $q(t)$:

$$q(t) = \sum_{n=1}^{N} q_n(t) \quad . \tag{II.8.8}$$

Then the spectral density of the sum is N times the spectral density of each,

$$S_q(\omega) = N\, S_{q_n}(\omega) \quad ,$$

as noted in the previous subsection. Therefore, the spectral density of force applied to the mass (Eq. II.8.7) can be written

$$S_f(\omega) = S_q(\omega) \langle |H_n|^2 \rangle_n \qquad \text{(II.8.9)}$$

where the quantity in brackets is a simple average over the different points of application of the squares of transfer functions:

$$\langle |H_n|^2 \rangle_n = \frac{1}{N} \sum_{n=1}^{N} |H_n|^2 \quad .$$

It follows that the spectral density of response is

$$S_v(\omega) = |Y(\omega)|^2\, S_q(\omega)\, \langle |H_n|^2 \rangle_n \quad . \qquad \text{(II.8.10)}$$

The mean-square response, got by integration over frequency, can be written

$$\langle v^2 \rangle_t = \langle v_q^2 \rangle_t \, \langle |H_n|^2 \rangle_n \quad , \qquad \text{(II.8.11)}$$

where $v_q(t)$ would be the response if the total force $q(t)$ were applied directly, with a unit value of transfer function.

Ensemble average: These results are capable of another interpretation. Suppose that a single force, equal to the total $q(t)$, is applied to one position z_n on the lever. Repeat the experiment, applying the same force to different points, until the force has been applied to each point for the same number of times. What, then, is the average mean-square response, and what is the average spectral density? These averages are evidently given precisely by Eqs. II.8.10 and II.8.11.

Statistically speaking, the last paragraph describes an ensemble of experiments and asks for an ensemble average. Where the transfer function H_n varies from experiment to experiment, the effective value for the set of experiments is the ensemble average, taken in a mean-square sense:

$$H^2_{\text{effective}} = \langle |H_n|^2 \rangle_n \quad .$$

No question of cross-correlation arises in the ensemble of experiments since they are not run simultaneously.

Frequency-dependent coupling: In this sub-section, we have calculated the response of a single resonator to a number of random forces which have different strengths of coupling. Many practical situations are like this, differing only in the nature of the coupling. The present example -- coupling through a lever -- is not representative, because the strength of coupling is independent of frequency (Eq. II.8.5). However, the analysis for more general coupling would have proceeded in much the same way, except that frequency-dependent transfer functions must be introduced.

Both analytically and experimentally, the characteristics of frequency-dependent coupling are determined by the pure-tone response. Suppose that the n-th point of application of force is tested by a pure-tone force $q_n(t)$ which, in complex notation, has a complex amplitude Q_n. It will result in a pure-tone force on the mass, $f_n(t)$, which has a complex amplitude F_n. The ratio of complex amplitudes defines a complex transfer function

$$H_n(\omega) \equiv F_n/Q_n$$

which is, in general, a function of frequency.

Now we revert to the actual problem, in which the force applied to the n-th point is a complex signal characterized by its spectral density $S_{q_n}(\omega)$. It follows directly from the second fundamental theorem on spectral analysis (Eq. II.7.2 and subsequent discussion) that the spectral density of the resulting force on the mass is

$$S_{f_n}(\omega) = |H_n(\omega)|^2 \, S_{q_n}(\omega) \quad . \qquad \text{(II.8.12)}$$

This result is formally identical with Eq. II.8.6 except for substitution of the new, frequency-dependent transfer function. All the other equations relating spectral densities (Eqs. II.8.7, II.8.9, II.8.10) are similarly valid.

Only in the integration of the spectral density of response velocity to get the mean-square response, does the frequency dependence of the transfer functions affect the analysis. It is instructive to trace out this effect for the case of a single broad-spectrum force, $q_n(t)$.

The novel effect of frequency-dependent coupling is seen in Eq. II.8.12 to be a modification of the shape of the force spectrum, as a function of frequency. Frequency-independent coupling only modifies the overall magnitude, without change in shape. In many cases of interest, the change in spectrum shape has little effect on the response. This is the case when the variations of the transfer function are small over the narrow bandwidth of the resonator, and when also there is no abnormally large peak in the transfer function. Under these common circumstances, only the value of the transfer function at the resonance frequency is important.

The analytical expression of these remarks follows. The mean-square velocity in response to a force with the spectral density given in Eq. II.8.12 is (see Eq. II.7.4)

$$\langle v^2 \rangle_t = \int |H_n(\omega)|^2 \, S_{q_n}(\omega) \, |Y(\omega)|^2 \, d\omega \quad .$$

When the product $|H_n|^2 S_{q_n}$ varies but slightly in the bandwidth of admittance Y and has no abnormal peaks, it can be treated as a constant with the value that obtains at the resonance frequency ω_o. (Compare the identical procedure in the case of a broad-spectrum force with unity coupling, Eq. II.7.6.) Then, the frequency-dependent coupling can be treated as a frequency-independent coupling with a constant transfer function $H_n(\omega_o)$. The rest of the analysis proceeds as before.

II.8.c Pure-tone Forces with Various Frequencies

A final example of multiple excitation is a resonator excited directly by a set of pure-tone forces of different frequencies, all of which have the same rms value. An individual force is denoted

$$f_n(t) = \sqrt{2}\, f_o \cos(\omega_n t + \phi_n) \qquad \text{(II.8.13)}$$

where the values of ω_n and ϕ_n are different for each force, and f_o is the common rms value of force.

Now consider the set of forces with different frequencies. If two forces are applied simultaneously, their cross-correlation vanishes; the averages over all time t,

$$<f_1(t)f_2(t+\tau)>_t \propto <\cos(\omega_1 t+\phi_1)\ \cos(\omega_2 t+\omega_2\tau+\phi_2)>_t \quad ,$$

vanish for all values of τ, as long as ω_1 and ω_2 are distinct. Therefore, the cross-spectral densities vanish and the spectral densities simply add (Eq. II.8.4). Evidently, it is unnecessary to distinguish between experiments where the forces are applied simultaneously and those where they are applied individually; the mean-square response will be the same in each case.

We consider here a particular ensemble average problem, in which the forces are applied individually at a large number of frequencies distributed uniformly in the band between ω_1 and ω_2. The response to any one force is given by the pure-tone relation

$$<v_n^2(t)>_t = f_o^2\ |Y(\omega_n)|^2 \quad .$$

The average mean-square response for all the forces is similarly related to the average of $|Y|^2$. Because the frequencies ω_n are uniformly spaced, the average over discrete frequencies converges to a continuous average as the spacing is reduced, i.e. the number of different frequencies is increased. The average response becomes

$$\begin{aligned} <v_n^2(t)>_{t,\omega_n} &= f_o^2 <|Y(\omega_n)|^2>_{\omega_n} \\ &= \frac{f_o^2}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} |Y|^2\ d\omega \quad , \end{aligned} \qquad \text{(II.8.14)}$$

where the response is averaged not only in time but also in frequency.

Comparison of this result with the previous results for broad-band excitation (e.g. Eq. II.7.4) shows that

$$f_o^2/(\omega_2-\omega_1)$$

plays a role identical to that of the spectral density $S_f(\omega)$ in the previous case. We see that there is an intimate relationship between the response to a band of noise and the average response to pure tones in the band, such as might be determined by slowly sweeping the frequency of a pure-tone force through the band.

II.8.d Multiple Resonators: Approximate Formulation

There was only one resonator in each of the previous examples, which differed in the character of the forces causing motion. We now turn to a case involving a large number of resonators with different characteristics: natural frequencies, mass, damping, etc. However, we assume that a single force is the ultimate source of vibration for each resonator.

It is well known that the general vibration of a large, complicated structure can be described in terms of the superposition of responses in its various natural modes, each of which has a different resonance frequency. A single broad-spectrum force will excite many modes. In fact, it is often difficult to rig an experiment in such a way that response is limited to a single natural mode. Except at frequencies near the lowest or fundamental resonance, even pure-tone forces will generally excite significant response in several modes. The response of each mode is governed by the same laws as the response of a single resonator.

However, we will not now consider the manner in which one models the vibration of a particular complicated structure by the vibration of a set of simple resonators. The details of modelling depend very much upon the structure's design. Specific cases will be taken up in later chapters. Instead, we continue to describe each resonator as a single mass-spring-dashpot combination.

A set of many independent, lightly-damped simple resonators are driven simultaneously by the same force f(t). Two of the multitude are sketched in Fig. II.6. Each resonator is driven through a coupling system with its individual transfer function. These are indicated in the figure by levers, but the analysis will treat the more general, frequency-dependent coupling which was discussed earlier (section 8b). Thus, the spectral density of the force applied to the mass of the k-th resonator is

$$S_{f_k}(\omega) = |H_k(\omega)|^2 \, S_f(\omega) \qquad \text{(II.9.1)}$$

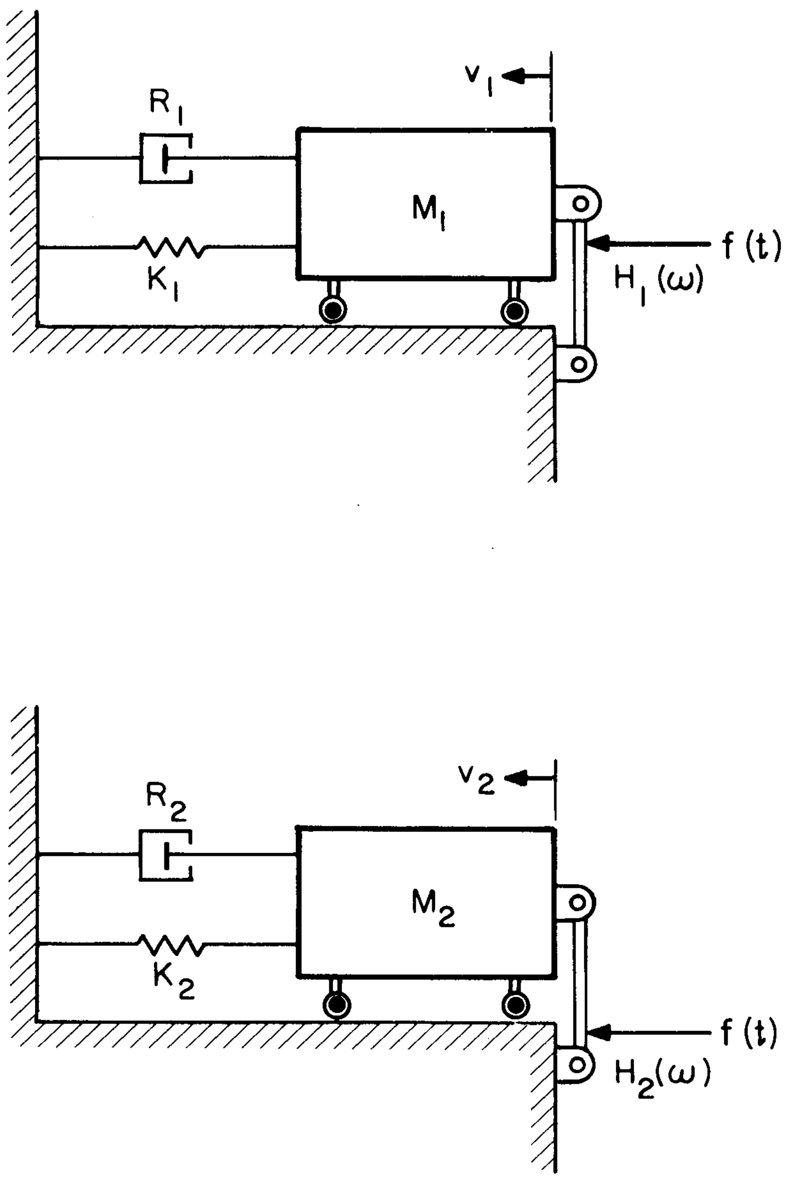

Figure II.6.- Two of a set of different resonators driven by the same force.

where H_k is the transfer function for the k-th resonator and S_f is the spectral density of the common force f(t). (Compare Eq. II.8.12.) We assume that the transfer functions are slowly varying functions of frequency.

Each resonator has a different resonance frequency ω_k and can have different values of mass, stiffness, and resistance. Since each resonator is independent of the others, its response can be independently determined from its admittance and the spectral density of applied force in the manner set forth previously in section 6.

<u>The force spectrum</u>: We wish to find the response of the whole set of resonators (e.g., the total vibratory energy) when it is driven by a broad-spectrum random force. Of course, the response is different for different forces. In order to get a generally useful result, we shall analyze the response for a limited band of frequency. The spectrum of any force can be characterized by its values in adjacent, limited bands of frequency. The response spectrum in any band is proportional to the strength of the force in the same band. Thus, an analysis band-by-band yields results applicable to any broad-spectrum force.

The typical force that we consider is a band-limited force which has a flat spectral density S_f within a limited range of frequency, specifically a band of width W centered on ω_c. Outside this band the spectral density is zero (Fig. II.7).

A moderately narrow bandwidth is desirable in order to reveal gradual trends in the response as the center frequency is changed. However, if the bandwidth is too narrow, we can expect the response to fluctuate rapidly with changes in frequency, corresponding to excitation at resonance of individual resonators. Therefore, we restrict the force's bandwidth W to being moderately wide, in the following two senses:

i) W is large enough to contain many resonance frequencies, which pertain to a "typical" selection of resonators;

ii) W is large compared with the effective bandwidth of each resonator, $\Delta_k = \frac{1}{2}\pi\eta_k\omega_k$.

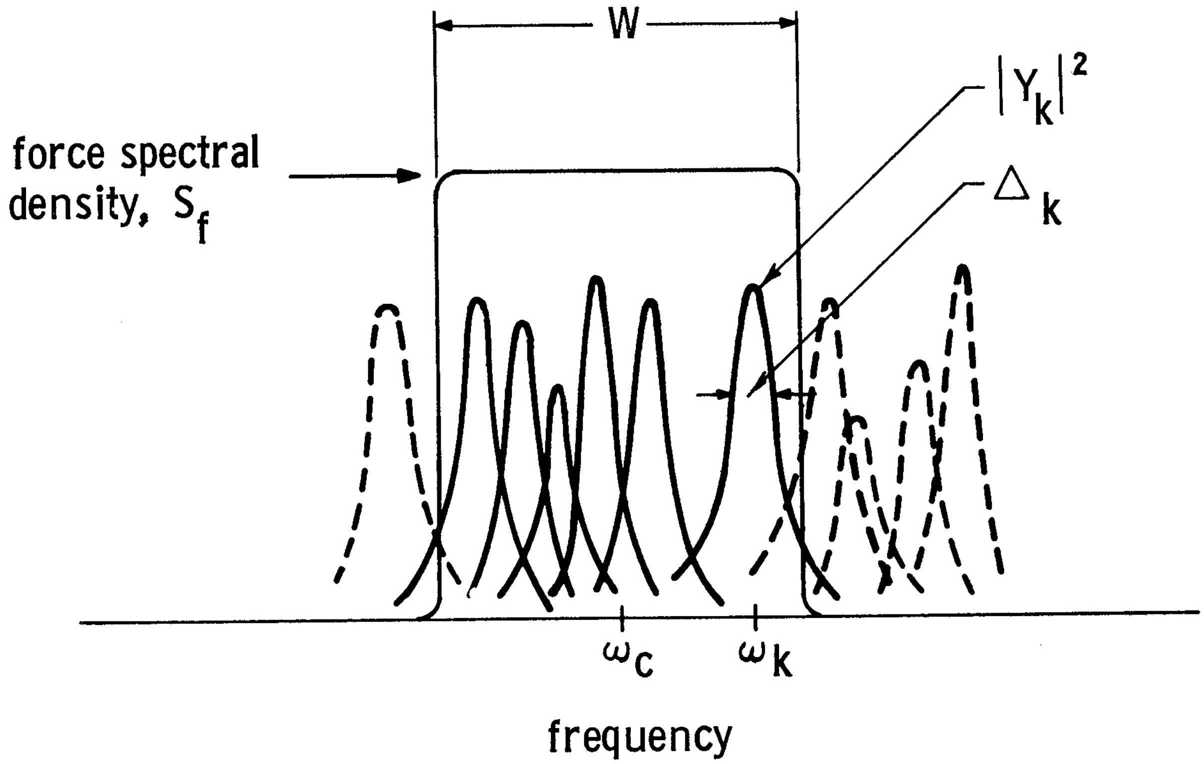

Figure II.7.- A band of force and the resonance curves of the resonators that it drives.

The case illustrated in Fig. II.7 is wide in the second sense, for $W \approx 8\Delta_k$. Moreover, it contains 6 resonance frequencies, which is often found to be wide enough in the first sense. In typical practical cases these restrictions are not very severe.

Calculation procedure: The response of any one resonator is large when its resonance frequency lies in the band of force and small when it lies outside. Since there are many resonances in W, we need not worry too much about borderline cases where a resonance lies exactly on the edge of W. (Experimentalists will know that it is unrealistic even to speak of an exact edge to a band-limited force.) We propose to use the following simple calculation scheme:

i) If ω_k is not in W, that resonator's response is assumed to be zero;

ii) if ω_k is in W, that resonator's response will be calculated as if the force spectrum were flat and W were "infinitely" broad.

Counteracting errors are produced by these two approximations. The first ignores the non-resonant response of all resonators outside W, and also the response in borderline cases. This leads to underestimation of total response. The second overestimates the response of those resonators with ω_k in W. The net error, and whether it be an overestimation or underestimation, cannot be determined simply. Brute force calculation could be used to check specific cases. A more general and reliable check is afforded by comparison of calculations and experimental results; the simple calculation scheme has met this test on numerous occasions, with accuracy sufficient for good engineering predictions.

The approximation most likely to lead to grief is the neglect of non-resonant response. The net forces acting on different masses are all different, because of the differing transfer functions H_k. (See Eq. II.9.1.) It is conceivable that numerous non-resonant resonators can be strongly excited, because they have large H_k, while the resonant ones with ω_k in the force band have small H_k and are lightly excited. The total non-resonant response could exceed the resonant response in such biased circumstances. Only a few physical situations of this sort have been found.

As a precautionary measure, it is desirable to complement calculations of the sort we are about to perform by separate calculations of the total non-resonant response, i.e. the response in the band of all resonators whose resonance frequency lies outside the band. It is usually a fairly simple task to estimate the total magnitude of non-resonant response, since it is independent of damping and relatively insensitive to the precise locations of resonance frequencies. However, for the present, let us accept the approximation inherent in the simple calculation scheme and proceed with the arithmetic.

Analysis: We wish to find the total response of the set of simple resonators, all driven by the same band-limited force with spectral density S_f. We shall find two characteristics of total response: the total energy, which is the sum of energies of individual resonators, and the total power dissipated, which is the sum of the individual powers. According to the simple approximation, the sums need only include resonators having their resonance in the force band W.

Consider a typical resonator, the k-th, with ω_k in the band W. Its mean square velocity is (Eq. II.7.6).

$$\begin{aligned} \langle v_k^2(t)\rangle_t &= \tfrac{1}{2}\pi S_{f_k}(\omega_k)/R_k M_k \\ &= \tfrac{1}{2}\pi S_f |H_k(\omega_k)|^2/R_k M_k \quad , \end{aligned}$$

where we use Eq. II.9.1. Its time-averaged energy and power dissipated (Eq. II.7.11) are

$$\langle E_k\rangle = M_k\langle v_k^2\rangle$$

$$\langle \Pi_k\rangle = R_k\langle v_k^2\rangle \quad .$$

The total energy or power for the whole set of resonators is the sum of the E_k or Π_k for the excited resonators alone.

Great convenience results from expressing this sum as an average. At the same time we introduce the mean-square value of the force (cf. Eq. II.7.1)

$$\langle f^2\rangle = \int_{\omega_c - \frac{1}{2}W}^{\omega_c + \frac{1}{2}W} S_f \, d\omega = S_f \, W \quad .$$

Then, if N denotes the total number of resonators with ω_k in W, the sum for total energy can be written

$$\langle E\rangle_t = \sum_{(\omega_k \text{ in } W)} \langle E_k\rangle_t = \tfrac{1}{2}\pi\langle f^2\rangle_t \frac{N}{W}\left[\frac{\Sigma(|H_k|^2/R_k)}{N}\right] .$$

The last term on the right is an average over those resonators whose natural frequencies lie in the band:

$$N^{-1}\Sigma(|H_k|^2/R_k) = \langle|H_k|^2/R_k\rangle_{(\omega_k \text{ in } W)} .$$

The ratio N/W equals the number of resonance frequencies in W per unit of frequency; we call it the <u>modal density</u>:

$$n(\omega_c, W) \equiv N/W \quad . \tag{II.9.2}$$

Its reciprocal represents the average frequency interval between resonances.

In summary, we have found the response of a whole set of resonators to excitation by a common force whose spectrum is limited to a band W including only some of the resonance frequencies. The time averages of total energy and total power are expressible in terms of the mean-square force and average characteristics of the resonators:

$$\langle E\rangle_t = \tfrac{1}{2}\pi\langle f^2\rangle_t \; n(\omega_c, W)\langle|H_k|^2/R_k\rangle_{(\omega_k \text{ in } W)} \tag{II.9.3}$$

$$\langle \Pi\rangle_t = \tfrac{1}{2}\pi\langle f^2\rangle_t \; n(\omega_c, W)\langle|H_k|^2/M_k\rangle_{(\omega_k \text{ in } W)} \tag{II.9.4}$$

<u>Discussion</u>: Since the dynamics of a collection of lever-driven trolleys is hardly a topic of compelling interest, the calculations given above may seem to be but trivial manipulations of routine formulas. In a sense, the conclusion is valid, for the manipulations <u>are</u> quite simple. However the results are important. Formulas of this sort form the basis for predicting the response to noise of actual multi-modal structures, as will be seen in later chapters. There, it is shown that such average quantities as modal density and $\langle|H_k|^2/R_k\rangle$ are:

i) relatively independent of the bandwidth W, if it be not too small;

ii) slowly varying functions of the center frequency ω_c; and

iii) in many cases insensitive to details of construction such as may determine the precise value of resonance frequencies.

The rest of this study will in large part be concerned with the development of a modal description of structural vibration and the determination of analytical expressions for the average quantities in particular cases.

II.9 Multiple Resonators: Precise Formulation

In the preceding approximate formulation of the multiple resonator problem, the accuracy of the approximations was questioned. No numerical answer was given, as that is impossible until the exact details of the system are specified. Instead, reliance was placed on the tendencies for error cancellation and on existing validations by experiment. Such an answer is not entirely satisfying, especially since it does not help one to anticipate difficulties in new situations.

However an evaluation of errors which is only pertinent to some one precisely determined system is not at all to our liking for several reasons. We are forced to ask for information that is unavailable in most practical situations. We have no reason to believe the answer to be typical of all similar structures. Finally, that answer will change rapidly with slow changes in the center frequency ω_c and bandwidth W, because the modal density (Eq. II.9.2) will undergo discrete, stepwise changes.

An escape from this dilemma is afforded us by the statistical trick of "ensemble averaging." Instead of analyzing a single set of resonators, we consider an infinite ensemble of sets, each having similar but slightly different characteristics. We shall then find it possible to derive a precise value for response which is an average, taken across the ensemble.

Of course, the answer found in this way is not that which we started to look for; the ensemble average of response need not equal the actual response of any individual set of resonators.

Yet, in many respects, the ensemble average is more nearly the answer we _should_ have been looking for; it was self-deception to believe that we could predict, for example, the exact resonance frequencies of a complicated structure.*

Thus, the ensemble averaging technique is an analytical method for eliminating undesired fluctuations from the calculations. In the present case, these fluctuations arise from the fact that a single set of resonators has only a finite modal density. We proceed to illustrate the technique for a special case in which every resonator has the same values of mass, loss factor, and transfer function.

Sets of resonators; ensembles: The word _set_ will denote a single collection of simple resonators. The single set is identified by one index number, α in general, and the single resonator within the set is identified by an additional index number, k in general. The resonance frequencies $\omega_{\alpha k}$ of resonators in the α-th set range from very low to indefinitely high (i.e., to "infinity"). We assume that every resonator has the same mass M, loss factor η, and the same constant transfer function $|H|$.

The _ensemble_ is an infinite collection of sets, each differing slightly in characteristics. As a practical example, a single set of resonators may represent the bending modes of a particular flat plate. A different set may represent another plate of the same area but with slightly different dimensions. The ensemble, then, represents all plates of the same area.

Ensemble averages: We are interested in computing the ensemble average of the total response of sets of resonators, when every set is excited by the same force. We could, then, start with the familiar formulas for one resonator, add up the results for all resonators of a set, and average the sums. For example, let $E_{\alpha k}$ represent the total energy of the (αk)-th resonator, whose resonance frequency is $\omega_{\alpha k}$. Then the sum

*Moreover, having once determined a precise answer for the ensemble average, we have a basis for examining the variations from this average that are to be expected in individual structures. See R. H. Lyon and E. Eichler, "Random Vibration of Connected Structures", JASA _36_; No. 7, pp. 1344-1354, July 1964.

$$E_\alpha = \Sigma_{k=1}^{\infty} E_{\alpha k} \tag{II.10.1}$$

is the total energy of response of the α-th set of resonators. The ensemble average total energy is

$$\langle E_\alpha \rangle_\alpha = \lim_{A \to \infty} \frac{1}{A} \Sigma_{\alpha=1}^{A} E_\alpha \quad . \tag{II.10.2}$$

This average involves a double summation over the energies of individual resonators:

$$\langle E_\alpha \rangle_\alpha = \lim_{A \to \infty} \frac{1}{A} \Sigma_{\alpha=1}^{A} \Sigma_{k=1}^{\infty} E_{\alpha k} \quad . \tag{II.10.3}$$

Now, it is much more convenient to perform these sums in a different order, specifically according to the magnitude of the resonance frequencies $\omega_{\alpha k}$. In this new counting procedure, we first add the contributions of all resonators of the ensemble having $\omega_{\alpha k}$ in a small frequency band δ centered on some frequency ω_o. Secondly, we take the ensemble average of these sums, as in Eq. II.10.2. Finally, we add the contributions from all such frequency bands. The result is $\langle E_\alpha \rangle_\alpha$.

Consider the second step in more detail. We assumed at the start that each resonator has the same mass, loss factor, and transfer function. Then all resonators with natural frequencies in the same small band δ will also have nearly identical values of $\omega_{\alpha k}$, of $R_{\alpha k} = \eta \omega_{\alpha k} M$, and of $K_{\alpha k} = \omega_{\alpha k}^2 M$. In brief, they are dynamically nearly identical and their separate responses will be indistinguishable if δ is small enough. The double sum in Eq. II.10.3 can then be replaced by the response of one typical resonator with natural frequency at the center of the band, ω_o, multiplied by a factor equal to the total number of resonators in the band. In the ensemble average, the factor becomes the ensemble average of the number of resonances that an individual set has in the band δ centered on ω_o. In the notation of Eq. II.9.2, the α-th set has

$$\delta \cdot n_\alpha(\omega_o, \delta)$$

resonances in the band. Therefore the contribution of the band to $\langle E_\alpha \rangle_\alpha$ can be written

$$E_o(\omega_o)\ \delta\ \langle n_\alpha(\omega_o, \delta) \rangle_\alpha$$

where E_o is the energy of one typical resonator with natural frequency ω_o.

Now, the third step of the modified summing procedure is to add the contributions from all frequency bands, each having a small width δ but a different center frequency ω_o. By taking more and more bands of smaller and smaller widths, this sum becomes equivalent to an integration over the continuous frequency variable ω_o. In this limit, $\langle n_\alpha \rangle_\alpha$ becomes a continuous function of frequency,* the ensemble-average modal density:

$$\overline{n}(\omega_o) = \lim_{\delta \to o} \langle n_\alpha(\omega_o, \delta) \rangle_\alpha \quad . \tag{II.10.4}$$

The average number of resonances that a single set has in a small band δ centered on ω_o is $\overline{n}(\omega_o) \cdot \delta$. The ensemble average of total energy in a set can now be written as an integral over resonance frequency

$$\langle E_\alpha \rangle_\alpha = \int_{\omega_o=0}^{\infty} E_o(\omega_o)\ \overline{n}(\omega_o) d\omega_o \tag{II.10.5}$$

where $E_o(\omega_o)$ is the total energy of a single resonator with natural frequency ω_o. This integral formulation is much more convenient for examining the average response than the double summation of Eq. II.10.3, although the two expressions are completely equivalent.

A moment's reflection shows that the equivalence is widely applicable and not at all restricted to total energy, although the derivation was phrased in those terms. Let $E_{\alpha k}$ stand for any characteristic of the response of an individual resonator: total energy, mean-square velocity, spectral density, etc. Let E_α stand for the sum over all resonators of the α-th set. Then the ensemble average $\langle E_\alpha \rangle_\alpha$ is given by the integral of Eq. II.10.5. (The essential restriction, that the response of an individual resonator depend only on its resonance frequency, is satisfied because of the assumption that each resonator has the same mass, loss factor, and transfer function.) We proceed to use Eq. II.10.5 in calculating some average spectral densities.

*A mathematician may well object to calling this a continuous function. Without specifying the ensemble in detail, we cannot be sure that $\overline{n}$ does not have singularities, such as would occur if one resonator of every set had identically the same natural frequency. At worst, $\overline{n}$ may be a generalized function with singularities of the δ-function type.

Spectral density of energy: The spectral density, at frequency ω, of the response velocity of a single resonator with natural frequency ω_o can be written (Eqs. II.6.11, II.7.2a, II.7.2b):

$$S_v(\omega;\omega_o) = S_f(\omega)(|H|^2/M^2)\ y^2(\omega;\omega_o) \qquad \text{(II.10.6)}$$

where

$$y^{-2}(\omega;\omega_o) = \omega_o^2\left[\eta^2 + \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2\right] \quad .$$

Here $S_f(\omega)$ is the spectral density at ω of the force. The spectral density of the single resonator's acceleration or displacement are obtained from S_v by multiplying or dividing by ω^2 (Eq. II.7.3).

We now introduce some new concepts, the spectral densities of energy functions. Consider, for example, the kinetic energy T of the single resonator.* It is readily written as

$$T = \tfrac{1}{2}M\langle v^2\rangle_t = \int_{\omega=0}^{\infty}\left[\tfrac{1}{2}M\ S_v(\omega)\right]d\omega \quad .$$

The integrand in brackets is the spectral density of kinetic energy, representing the contribution to T of response components with frequencies near ω. We will denote it by S_T. Spectral densities for potential energy U and for power Π are defined analogously. For the single resonator with natural frequency ω_o, their expressions are found by familiar relationships to be:

$$S_T(\omega;\omega_o) = \frac{1}{2}\ M\ S_v(\omega;\omega_o)$$

$$S_U(\omega;\omega_o) = \frac{1}{2}\ K\ S_x(\omega;\omega_o) = \tfrac{1}{2}(\omega_o^2/\omega^2)M\ S_v(\omega;\omega_o) \qquad \text{(II.10.7)}$$

$$S_\Pi(\omega;\omega_o) = R\ S_v(\omega;\omega_o) = \eta\omega_o\ M\ S_v(\omega;\omega_o)$$

where S_v is given by Eq. II.10.6.

*Throughout this section, we are concerned with time-averages of the energy functions, although the average with respect to time is not explicitly indicated.

The values of the energy functions (T, U, and Π) for a whole set of resonators are equal to the sums of contributions from all resonators of the set. The same is true for the spectral densities of the energy functions. Therefore we may use the general formula, Eq. II.10.5, to compute their ensemble averages.

Consider the potential energy first. The application of Eq. II.10.5 to S_U, as given above, yields the ensemble average value of the spectral density of a set's total potential energy:

$$\langle S_U(\omega)\rangle_\alpha = \int_0^\infty S_U(\omega;\omega_o)\ \bar{n}(\omega_o)\ d\omega_o$$

$$= \frac{1}{2}\ S_f(\omega)\int_0^\infty \frac{|H|^2}{M}\ \frac{\omega_o^2}{\omega^2}\ y^2(\omega;\omega_o)\ \bar{n}(\omega_o)\ d\omega_o \quad . \quad \text{(II.10.8)}$$

Note that this integral is a function only of ω; it reflects the contributions of all resonators in a set but its value does not involve any particular resonance frequency. At the same time, the integral does not depend on the characteristics of the exciting force, which appear only in the factor S_f. By introducing the spectral density of the energy and by taking an ensemble average, we have been able to develop a general but conceptually simple formula of the type:

(response) = (force) x (response function).

Of course, the arithmetic may get complicated in particular instances.

Constant modal density: An instance of great practical importance is that where $\bar{n}(\omega_o)$ is a constant: in the ensemble average, the natural frequencies are equally spaced, although any one set of resonators will exhibit fluctuations about the average. As we shall see later, the natural frequencies of bending modes in a flat plate are equally spaced, on the average. In this case, Eq. II.10.8 becomes

$$\langle S_U(\omega)\rangle_\alpha = \frac{1}{2}\ S_f(\omega)\frac{|H|^2}{M}\ \bar{n}\int_0^\infty \frac{\omega_o^2}{\omega^2}\ y^2(\omega;\omega_o)\ d\omega_o \quad . \qquad \text{(II.10.9)}$$

The integral in this equation is formally identical with that required to obtain the mean-square velocity of a single resonator excited by a broad-spectrum force (Eq. II.7.5) The similarity is more than a coincidence, although the physical processes are quite different. There, we were adding the many different spectral components of the response of one resonator. Here we add the spectral components at one frequency, ω, of many different resonators. The value of the integral is

$$\int_0^\infty \frac{\omega_o^2}{\omega^2} y^2(\omega;\omega_o)\ d\omega_o = \pi/2\eta\omega \quad , \tag{II.10.10}$$

whence it follows that

$$\langle S_U(\omega)\rangle_\alpha = \tfrac{1}{4}\pi\ S_f(\omega)\ \frac{\overline{n}|H|^2}{\eta\omega M} \quad . \tag{II.10.11}$$

A similar analysis can be carried through for the kinetic energy, using the expression for S_T given in Eqs. II.10.8. The details will not be reproduced here. In the case of constant modal density, the ensemble-averaged spectral density of a set's kinetic energy, $\langle S_T(\omega)\rangle_\alpha$, is the same as Eq. II.10.9 except for the integral. The new integral

$$\int_0^\infty y^2(\omega;\omega_o)\ d\omega_o \tag{II.10.12}$$

has the same total value as the old, although it gives more weight to resonance frequencies ω_o less than ω and less weight to values greater than ω.* The result

$$\langle S_T(\omega)\rangle_\alpha = \langle S_U(\omega)\rangle_\alpha \tag{II.10.13}$$

is exactly true at every frequency, for any force, and for any value of η.

The simplicity of these relations makes it possible to find a simple but precise expression for the total energy of a set of resonators driven by a force whose spectrum is limited to a band. The total energy E is the sum of total kinetic and potential energies. Each of these is given by the integral of its spectral density, and the two densities have been found to be equal. Thus, we compute the ensemble average

*The integral is formally identical with that required to get the mean-square displacement of a single resonator (Eq. II.7.7).

$$\langle E\rangle_\alpha = \langle T\rangle_\alpha + \langle U\rangle_\alpha = 2\langle T\rangle_\alpha = 2\int_0^\infty \langle S_T(\omega)\rangle_\alpha d\omega$$
$$= \tfrac{1}{2}\pi\ \bar{n}\ \frac{|H|^2}{\eta M}\int_0^\infty \frac{S_f(\omega)}{\omega}\ d\omega \quad .$$

But the rms force is given by the familiar relation

$$\langle f^2\rangle_t = \int_0^\infty S_f(\omega)\ d\omega \quad .$$

For a band-limited force, we can therefore write

$$\int_0^\infty \frac{S_f(\omega)}{\omega}\ d\omega = \frac{1}{\omega_c}\int_0^\infty S_f(\omega)\ d\omega = \langle f^2\rangle_t/\omega_c \quad ,$$

where ω_c is a "center" frequency lying somewhere in the band.* The final result for the total energy can thus be written

$$\langle E\rangle_\alpha = \tfrac{1}{2}\pi\ \langle f^2\rangle_t\ \bar{n}|H|^2/R_c \quad , \qquad R_c = \eta\omega_c M \quad , \qquad \text{(II.10.14)}$$

where R_c is the resistance of a typical resonator which resonates at the center frequency of the force band. Aside from differences in notation, this ensemble-average of response energy is identical with the previous approximate expression, Eq. II.9.3!

The corresponding analysis for total dissipated power Π starts from the spectral density $S_\Pi(\omega;\omega_o)$ given in Eqs. II.10.7. The results of straightforward calculation will be given without proof. In the ensemble average, the spectral density of power dissipated in a set of resonators is approximately

* The exact value of ω_c depends on the shape of S_f. That it lies somewhere in the force band follows from the mean value theorem for integrals.

**The integral is reduced simply to a standard form by transforming to a new variable $x \equiv (\omega_o^2-\omega^2)$.

$$\langle S_{\Pi}(\omega)\rangle_{\alpha} \approx \tfrac{1}{2}\pi \; S_f(\omega) \; \bar{n}|H|^2/M$$

for small damping ($\eta \ll \pi$). Therefore the total power, found by integrating over all ω, is

$$\langle \Pi \rangle_{\alpha} \approx \tfrac{1}{2}\pi \; \langle f^2 \rangle_t \; \bar{n}|H|^2/M \quad , \qquad \text{(II.10.15)}$$

a form equivalent to the previous approximate expression, Eq. II.9.4.

Discussion: Let us summarize the major conclusions at this point. The ensemble averaging process eliminates fluctuations in the total response of a set of resonators which result from the discreteness of the natural frequencies in the set. In consequence, it is possible to derive precise general expressions for the energies of response. We have evaluated these expressions for a particular system, whose generality is restricted by assumptions made for simplicity of analysis: we assumed that every resonator has the same values of M, η, and $|H|$, and that the modal density is independent of resonance frequency.

Subject to these restrictions, we have found that the results of the simple calculation scheme (Eqs. II.9.3 and II.9.4) are correct. That scheme was based on the neglect of any resonator whose natural frequency lies outside the force band, and on an approximate calculation for the response of the rest of them. That these approximations should yield precise ensemble averages in the case of some restricted systems is a reassuring conclusion. It suggests that the simple scheme may yield good approximations in less restricted systems.

The ensemble-averaging approach can, of course, be applied to less restricted situations. For example, the extension to cases where the transfer function $|H|$ is not constant would be a routine, if tedious, exercise. The effect of slow variations in the ensemble-average modal density could also be investigated. However, these extensions are matters for future research.

III. SOUND WAVES

III.1 Introduction

For the purposes of this course, sound is a dynamic disturbance from equilibrium of the physical characteristics of a fluid. The local pressure and temperature of the fluid fluctuate from their steady values, and the position of any small (but macroscopic) fluid element fluctuates correspondingly. We shall require some familiarity with the laws governing the propagation of sound through the fluid medium and with the relationships between such measures of sound strength as the fluctuating pressure, the energy associated with the fluctuations, etc.

Sound generated in one region will propagate away, reflect from nearby walls or structures into different paths, and quickly redistribute itself in such complicated fashions as to harass unduly the analyst intent on describing it. Fortunately, several idealized cases are close enough to reality that predictions based on them are found to be pertinent to a wide variety of practical situations. These idealizations are "free space", unbounded and uncluttered by obstacles other than the one structure of interest, and the rectangular room. We shall study sound waves in these two idealized regions with some care.*

III.2 Sound Wave Equation

The first task is to develop the differential equations governing the propagation of sound. In so doing, it will be assumed that the fluctuations (of pressure, etc.) are small so that only the terms linear in the sound variables need be retained. Viscous forces are neglected.

Consider a fluid which is stationary in the absence of sound fluctuations; its physical state is described by the steady pressure P_o and density ρ_o, which are assumed to be everywhere the same. Focus attention on an element of the fluid so small that

*Detailed discussions of these and related topics will be found in various introductory texts on acoustics. For example, see P. M. Morse, Vibration and Sound (McGraw-Hill Book Company, New York, 1948); L. L. Beranek, Acoustics (McGraw-Hill Book Company, New York, 1954); L. E. Kinsler and A. R. Frey, Fundamentals of Acoustics (John Wiley and Sons, New York, 1962).

one can never detect variations of pressure or density within it; such an element is called a "particle" of the fluid. The static position of the particle, i.e. its position in the absence of sound, will be denoted in a rectangular cartesian coordinate system (x_1, x_2, x_3) by

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) \quad .$$

(We use subscripts to distinguish the different axes.)

In the presence of sound, the pressure density, and position of the particle will fluctuate from their static values:

$$\text{total pressure:} \quad P_o + p(\bar{x}_1, \bar{x}_2, \bar{x}_3, t)$$

$$\text{total density:} \quad \rho_o + \rho_1(\bar{x}_1, \bar{x}_2, \bar{x}_3, t)$$

$$\text{position:} \quad x_j = \bar{x}_j + \xi_j(\bar{x}_1, \bar{x}_2, \bar{x}_3, t) \quad , \quad j = 1,2,3.$$

(Note that the static position coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ are being used to identify the particle, even when it has moved to a different position.) The fluctuation in pressure p is called the (instantaneous) <u>sound pressure</u>; the change in position, with components ξ_j, is called the (instantaneous) <u>particle displacement</u>. The time derivative of particle displacement, with components

$$u_j = \partial\xi_j/\partial_t \equiv \dot{\xi}_j$$

is called the (instantaneous) <u>particle velocity</u>. (Note the use of a superscript dot to denote a partial derivative with respect to time.) The values of the functions p, ρ, and ξ_j or u_j can be related by three equations: Newton's law, conservation of mass, and the equation of state of the fluid. From these will come the wave equation for sound.

Consider a small rectangular volume element, composed of many "particles" whose static coordinates lie between the planes $\bar{x}_j + 1/2\delta\bar{x}_j$ and $\bar{x}_j - 1/2\delta\bar{x}_j$, where j = 1, 2, or 3 (Fig. III.1). In the dynamic situation, these particles will be displaced to varying degrees depending on their static position; the volume element defined by the particles will be changed in size and shape. For the present case of a fluid without viscosity, it

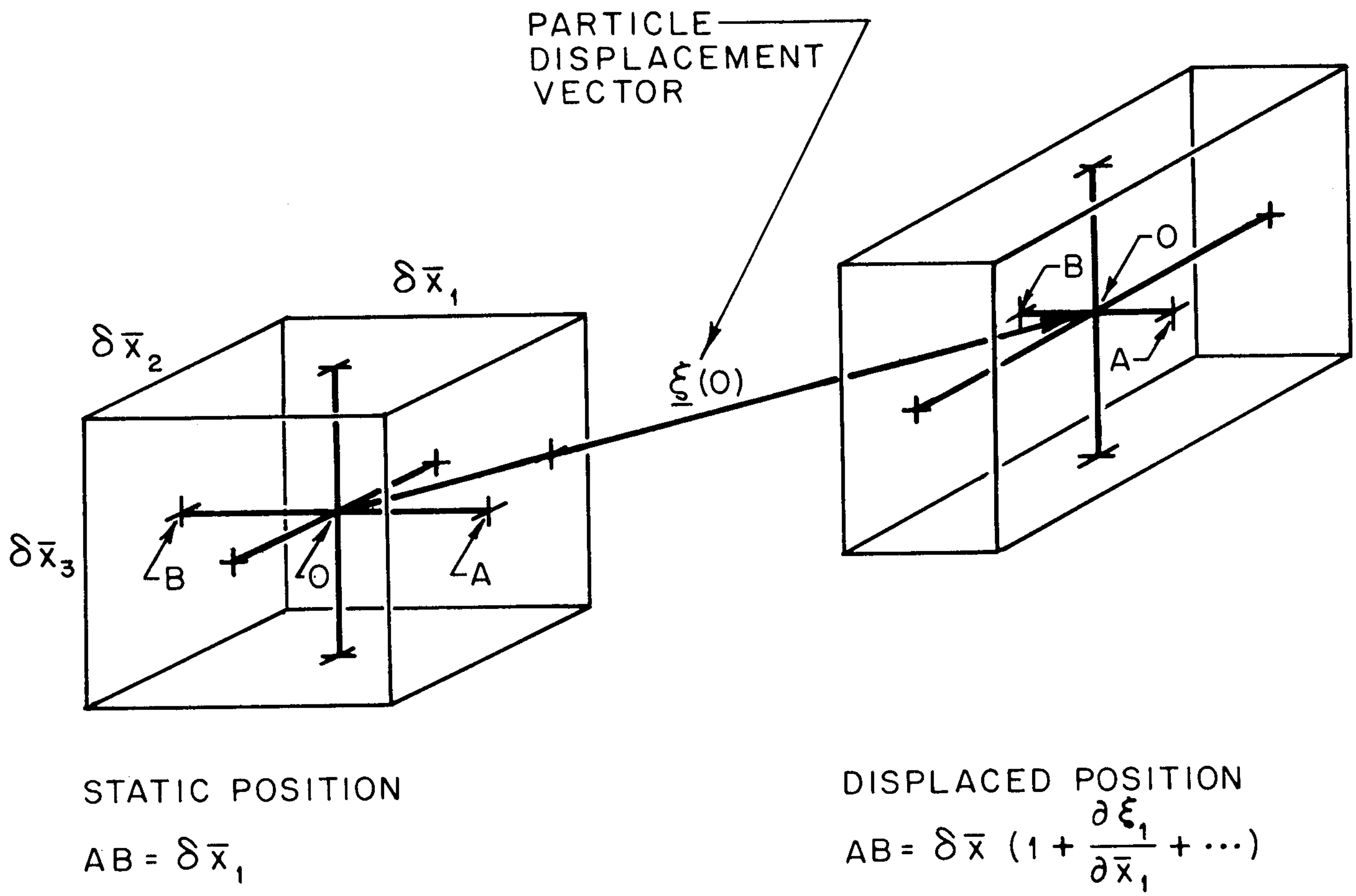

Figure III.1.- Displacement and distortion of a fluid volume element under the action of sound.

turns out that but a few of the characteristics of this dynamic distortion are required -- specifically the average change in particle position, and the change in volume of the element. It will also be necessary to evaluate the average pressure exerted on the various faces of the distorted element by the surrounding fluid.

The center particle of the volume element, point O in Fig. III.1, has coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and undergoes a displacement with components denoted by:

$$\xi_j(0) \qquad j = 1, 2, 3 .$$

Consider a typical nearby particle, with static coordinates $\bar{x}_k + \bar{\epsilon}_k$, $k = 1, 2, 3$. Its displacement is slightly different, given approximately by the first two terms of a Taylor's expansion:

$$\xi_j(\bar{x}_k + \bar{\epsilon}_k) = \xi_j(0) + \sum_{k=1}^{3} \bar{\epsilon}_k (\partial \xi_j / \partial \bar{x}_k) + \cdots .$$

It is evident from this formula that the average displacement of the volume element equals the displacement of the center particle:

$$\langle \xi_j \rangle = \xi_j(0) \quad .$$

Now, the static volume V_o of the element equals the product of the lengths of its three edges $\delta \bar{x}_j$ (i.e. the distances between opposite faces):

$$V_o = \delta \bar{x}_1 \; \delta \bar{x}_2 \; \delta \bar{x}_3 \quad .$$

In the distorted element the volume equals the product of the three <u>average</u> distances between opposite faces, for example, between the particles A and B in Fig. III.1. These distances are found from the Taylor's expansion to be

$$\delta \bar{x}_j [1 + (\partial \xi_j / \partial \bar{x}_j) + \cdots] \quad ,$$

so that the distorted volume equals

$$V_o + \delta V = V_o [1 + (\partial \xi_1 / \partial \bar{x}_1) + (\partial \xi_2 / \partial \bar{x}_2) + (\partial \xi_3 / \partial \bar{x}_3)]$$

to a first approximation in small quantities. The fractional decrease in volume is called the condensation and denoted by $s(\overline{x}_j)$; thus, the distorted volume is

$$V_o + \delta V = V_o(1-s) \tag{III.2.1}$$

where

$$-s = \partial\xi_1/\partial\overline{x}_1 + \partial\xi_2/\partial\overline{x}_2 + \partial\xi_3/\partial\overline{x}_3 \quad .$$

In vector notation, the right hand side is called the divergence of the displacement vector $\underline{\xi}$, and the equation is written

$$-s = \mathrm{div}\ \underline{\xi} \quad .$$

(Underscoring is used to indicate vectors in this report.)

In order to evaluate the average pressure exerted on the faces of the volume element, let the sound pressure be expanded in Taylor's series, in the same way as the displacement ξ was above. The sound pressure p at a point $\overline{x}_k + \overline{\epsilon}_k$ differs from the sound pressure at the center point $\overline{x}_k$ by a small amount:

$$p(\overline{x}_k+\overline{\epsilon}_k) = p(\overline{x}_k) + \sum_{k=1}^{3} \overline{\epsilon}_k(\partial p/\partial\overline{x}_k) + \cdots \quad .$$

Some simple analysis readily shows that the average pressure on any face equals the pressure at the center point of the face. Thus the average pressure on the face $x_1=\overline{x}_1+1/2\delta\overline{x}_1$ (particle A in Fig. III.1) is

$$p(A) = p(0) + \tfrac{1}{2}\delta\overline{x}_1(\partial_p/\partial\overline{x}_1) \quad .$$

We are now ready to fit these expressions into differential relations between the sound variables.

By Newton's law, the net force on the element due to the pressure of the external fluid is related to the acceleration. In the present case, the net force is the difference between pressures on opposite faces, multiplied by the area of the faces. For components in the x_1 direction, we find:

$$\{[P_o + p(\overline{x}_j) - \tfrac{1}{2}(\partial p/\partial \overline{x}_1)\delta\overline{x}_1] - P_o + p(\overline{x}_j) + \tfrac{1}{2}(\partial p/\partial \overline{x}_1)\delta\overline{x}_1]\}\delta\overline{x}_2\delta\overline{x}_3 =$$

$$(\rho_o\ \delta\overline{x}_1\delta\overline{x}_2\delta\overline{x}_3)\ \partial^2\xi_1/\partial t^2 \quad ,$$

or

$$\partial p/\partial\overline{x}_1 = -\ \rho_o\partial^2\xi_1/\partial t^2 \equiv -\ \rho_o\ddot{\xi}_1 \quad .$$

The equations for the two other coordinate directions are the same except for changes in the subscripts. The three equations together are called the force equation:

$$\partial p/\partial\overline{x}_j = -\ \rho_o\ddot{\xi}_j \quad , \qquad j = 1,2,3;$$

or, in vector notation:

$$\text{grad } p = -\ \rho_o\ddot{\underline{\xi}} = -\ \rho_o\dot{\underline{u}} \quad . \qquad \text{(III.2.2)}$$

(The vector operator written "grad ( )" is called the gradient; it yields a vector when applied to a scalar.)

A second fundamental equation results from the condition that the mass of this volume element is conserved in its motion. The fluid is neither created nor destroyed by the presence of sound; therefore volume changes are reflected in density changes. We equate the static mass (density times volume) and the mass in the dynamic position, using Eq. III.2.1:

$$\rho_o\ \delta\overline{x}_1\delta\overline{x}_2\delta\overline{x}_3 = (\rho_o+\rho_1)\delta\overline{x}_1\delta\overline{x}_2\delta\overline{x}_3(1-s) \quad .$$

Some further manipulation, retaining only first powers of ρ_1 and ξ_1 and dropping their products, leads to the continuity equation:

$$\rho_1 = \rho_o s = -\ \rho_o\ \text{div}\ \underline{\xi} \quad . \qquad \text{(III.2.3a)}$$

After this equation is differentiated with respect to time, one may introduce the particle velocity vector $\underline{u}$:

$$\dot{\rho}_1 = -\rho_o\ \text{div}\ \underline{u} \quad . \qquad \text{(III.2.3b)}$$

Hereafter, we shall drop the superscript bar in writing the coordinates, $\overline{x}_j$.

Here are two equations in the three unknown functions, sound pressure, particle displacement, and density fluctuation. A third equation is required, relating sound pressure and density fluctuations; it constitutes the dynamic equation of state. In ideal fluids, a small fluctuation in local pressure is reflected instantaneously by a proportional change in local density:

$$p = (K/\rho_o)\rho_1 \quad . \tag{III.2.4a}$$

The constant K is called the bulk modulus, and has the units of pressure. The ratio

$$c^2 \equiv p/\rho_1 = K/\rho_o \tag{III.2.4b}$$

is the square of the speed of sound, as will become evident in a moment.* In the case of ideal gases, when the sound fluctuations take place at such a moderate rate that there is no heat flux from the compressed gas ("adiabatic compression") the bulk modulus is related to the ambient pressure P_o:

$$K = \gamma \; P_o \quad ,$$

where γ is the ratio of specific heats of the gas. Then, the sound speed is

$$c = \sqrt{\gamma P_o/\rho_o} \quad ;$$

this is the usual formula for sound speed in a gas.

One may well anticipate that the real world does not always satisfy such simple relations. The constant proportionality of Eq. III.2.4 must sometimes be replaced by differential equations, involving derivatives with respect to time or to spatial position. Fortunately, the most common fluids, air and water, very closely satisfy the constant proportionality over a broad range of frequencies. We shall generally assume that the sound speed is a constant.

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*Sound "speed" is preferred to "velocity" in order to avoid confusion with the particle velocity.

The three equations must now be combined. If Eq. III.2.3 is differentiated twice with respect to time and the divergence operation is applied to Eq. III.2.2, the equations can be combined:

$$\text{div grad } p \equiv \nabla^2 p = -\rho_o \partial^2(\text{div } \underline{\xi})/\partial t^2 = \ddot{\rho}_1 \quad . \qquad \text{(III.2.5)}$$

(The combined operator "div grad ( )" or "∇^2( )" is called the Laplacian; applied to a scalar, it yields a scalar.) But the density fluctuation ρ_1 is proportional to the sound pressure p and can be eliminated. There results the wave equation for pressure:

$$\nabla^2 p - \ddot{p}/c^2 = 0 \quad . \qquad \text{(III.2.6)}$$

In cartesian coordinates the Laplacian can be written as a sum of derivatives:

$$\nabla^2 p = \Sigma_j \partial^2 p/\partial x_j^2$$

III.3 Plane Waves

Solutions of the wave equation for several special cases are particularly important. The first is the plane wave, in which the instantaneous sound pressure is constant over any plane perpendicular to some one straight line. Let x be the cartesian coordinate along the normal to the planes; then, the wave equation is simply

$$\partial^2 p/\partial x^2 - \partial^2 p/c^2\, \partial t^2 = 0 \quad . \qquad \text{(III.3.1)}$$

This equation has a very simple solution; the pressure is

$$p(x,t) = F(x-ct) + G(x+ct) \quad , \qquad \text{(III.3.2a)}$$

where F(y) and G(z) can have any functional dependence whatever and each is independent of the other.

The basis for interpreting the constant c, defined by Eq. III.2.4, as a speed of sound is now clear. The value of (x-ct) is constant at a point which moves in the direction of increasing values of x with a speed c. Therefore the whole pattern of pressure fluctuation in a sound wave of the type

F(x-ct) moves in the positive x direction with the speed c. Correspondingly a sound component of the type G(x+ct) represents a pattern of sound pressure which moves as a unit with speed c in the direct of decreasing values of x.

The particle velocity of the plane wave is readily found from the pressure by application of the force equation (III.2.2). Since p depends only on x, the gradient of p is a vector pointing in the x direction. The particle velocity vector has only an x component which must satisfy

$$\partial p/\partial x = -\rho_o \, \partial u/\partial t \quad .$$

But, from the form of Eq. III.3.2a, it follows that

$$\frac{\partial p}{\partial x} = -(1/c)\partial F/\partial t + (1/c)\partial G/\partial t \quad .$$

Since neither p nor u has any constant part, the combined equation can be integrated to yield

$$u = (1/\rho_o c)F - (1/\rho_o c)G \quad . \qquad \text{(III.3.2b)}$$

In the general case, a sound wave will undergo considerable distortion in its propagation away from a source and around obstacles. Plane waves represent convenient approximations to the behavior of sound in limited regions which are not too close to complicated boundaries nor to the source.

III.4 Pure-Tone Sound Waves

A second important special case is the steady, pure-tone sound field in which the sound pressure at any point is a simple-harmonic function of time. The analysis of pure-tone fields is most readily performed if the complex convention is used for the variables, in the manner explained in Chapter II, Section 5. The sound pressure variable at any point $\underline{r}$ is written as a complex quantity

$$p(\underline{r},t) = P(\underline{r})e^{i\omega t} \qquad \text{(III.4.1)}$$

(where ω is the frequency), it being understood that the real part of the complex quantity represents the physical variable. In general, $P(\underline{r})$ is a complex number whose modulus A and argument, or "phase", ϕ can both vary with position:

$$P(\underline{r}) = A(\underline{r})\ e^{i\phi(\underline{r})} \quad .$$

Thus, the physical pressure variable is the real part;

$$\mathrm{Re}(Pe^{i\omega t}) = A(\underline{r})\ \cos[\omega t+\phi(\underline{r})] \quad .$$

As noted in Chapter II, the complex variable satisfies the same differential equations as the real variable. However, time derivatives are more easily performed with the complex variable; a single time derivative is equivalent to multiplication by $i\omega$:

$$\partial p/\partial t = i\omega p \quad .$$

In the equations for pure-tone sound waves, the fluctuating density ρ_1 and the particle velocity vector $\underline{u}$ must also be denoted in the complex notation. Thus the complex particle velocity vector is

$$\underline{u}(\underline{r},t) = \underline{U}(\underline{r})\ e^{i\omega t}$$

where $\underline{U}(\underline{r})$ is a complex vector.

With these general matters of complex notation out of the way, we proceed to study the equations of sound propagation for pure tones. The fundamental differential equations are:

(i) the <u>force</u> <u>equation</u>,

$$\mathrm{grad}\ p = -\ i\omega\rho_o\underline{u} = -\ ik\rho_o c\underline{u} \quad , \qquad \text{(III.4.2)}$$

with $k \equiv \omega/c$ (compare Eq. III.2.2);

(ii) the combination of the <u>continuity</u> <u>and</u> <u>state</u> <u>equations</u>,

$$p = (i\rho_o c/k)\ \mathrm{div}\ \underline{u} \qquad \text{(III.4.3)}$$

(Compare Eqs. III.2.3B and III.2.4b); and

(iii) the <u>wave</u> <u>equation</u>*,

*To one mathematically inclined, this is a Helmholtz equation, not a "wave" equation, since it does not explicitly involve any time dependence.

$$\nabla^2 p + k^2 p = 0 \quad , \qquad k \equiv \omega/c \quad , \tag{III.4.4}$$

(Compare Eq. III.2.6).

The constant $k=\omega/c$ is called the wavenumber of the sound wave which has frequency ω (sometimes called the "phase constant" or "propagation constant"). The wavenumber plays a central role in acoustical analysis because it establishes the scale of the spatial variations of pressures.* In that regard it is analogous to the frequency, which establishes the scale of temporal variations since

$$\partial^2 p/\partial t^2 + \omega^2 p = 0 \quad .$$

Pure-tone plane wave: In the particular case of a pure-tone sound wave which is also plane, the solution for pressure becomes particularly simple. Let x be the coordinate along the direction normal to the planes of constant instantaneous sound pressure. The wave equation reduces to

$$\partial^2 p/\partial x^2 + k^2 p = 0$$

with a general complex solution

$$p(x,t) = A_1 e^{i(\omega t - kx)} + A_2 e^{i(\omega t + kx)} \tag{III.4.5}$$

where A_1 and A_2 are arbitrary complex constants.

---

*Note the form of the equations when k is used in writing the constants. So long as we are interested solely in pressures and velocities, k and the product $\rho_o c$ are the only constants. Moreover k always appears in association with a derivative operation of the same order, so that the k's disappear if the derivatives are rewritten as $\partial(\ )/\partial y_j$ with $y_j = kx_j$. The product $\rho_o c$ is another central parameter of acoustical analysis, carrying the formal name characteristic impedance but more usually called "rho-c". Its units are those of pressure divided by velocity.

Look at the first term alone. Its value is periodic in x, being the same at any two points x_a and x_b for which

$$(x_a - x_b) = n(2\pi/k) \quad ,$$

with n any integer. (The relation $\exp(i2\pi n)=1$ is valid for all integers n.) The fundamental period $(2\pi/k)$ is called the <u>wavelength</u>, denoted usually by λ:

$$k \equiv \omega/c \equiv 2\pi/\lambda \quad . \tag{III.4.6}$$

The second term is similarly periodic in x with a period equal to the wavelength λ, and thus the general solution is periodic.

The real pressure component corresponding to the first term of Eq. III.4.5 is

$$\mathrm{Re}\{p\} = B\cos(\omega t - kx + \phi) = B\cos[\phi - k(x-ct)] \quad ,$$

where we write $B\exp(i\phi)$ for the complex constant A_1. The functional dependence of this term is just that required of a wave travelling in the direction of increasing x with speed c (compare Eq. III.3.2a). Similarly, the second term corresponds to a wave travelling in the negative x direction.

These directional properties of a plane wave of single frequency are most concisely indicated by the further introduction of some simple vector analysis. Consider the factor kx in the first term of Eq. III.4.5. Let $\underline{\ell}$ be the unit vector oriented in the direction of increasing values of x, i.e., in the direction of propagation. We define a <u>vectorial wavenumber</u> $\underline{k}$ (or "k-vector") by associating the propagation direction $\underline{\ell}$, with the magnitude k:

$$\underline{k} \equiv k\underline{\ell} \quad .$$

Now, the position of any point on a plane, x=constant, can be described by a <u>position vector</u>

$$\underline{r} = x\underline{\ell} + \underline{z} \quad ,$$

where $\underline{z}$ is a vector lying in the plane. Thus, the vector $\underline{z}$ is perpendicular to $\underline{\ell}$. It follows that the phase factor $(-kx)$ can be written vectorially as a scalar product of vectors

$$-\,kx = -\,\underline{k}\cdot\underline{r} \quad .$$

In this vector form, the product is independent of the coordinates which are used to describe position.

The second term in Eq. III.4.5 was shown to be a wave travelling in the opposite direction (decreasing values of x), so that a unit vector in its direction of propagation is $\underline{\ell}' = -\underline{\ell}$. Correspondingly, its vectorial wavenumber would be $\underline{k}' = -\underline{k}$. We see that the phase factor (+kx) of this wave assumes a vectorial form

$$+kx = +\underline{k}\cdot\underline{r} = -\underline{k}'\cdot\underline{r} \quad ,$$

completely analogous to the first case.

In summary, any plane sound wave of frequency ω=kc travelling in the direction of the vector $\underline{k}$ constitutes a sound pressure

$$p(\underline{r},t) = A\, e^{i(\omega t - \underline{k}\cdot\underline{r})} \qquad \text{(III.4.7)}$$

in conventional complex, vectorial notation. The particle velocity associated with this wave is readily found from the pressure by Eq. III.4.2; in vector notation, the result is

$$\underline{u}(\underline{r},t) = \underline{\ell}\, p(\underline{r},t)/\rho_o c = \underline{\ell}(A/\rho_o c)\, e^{i(\omega t - \underline{k}\cdot\underline{r})} \quad , \qquad \text{(III.4.8)}$$

where $\underline{\ell} = \underline{k}/k$ is the unit vector in the direction of sound propagation. Note the appearance here of the constant $\rho_o c$, the <u>characteristic impedance</u>. The ratio of the pressure to the magnitude of the particle velocity vector equals $\rho_o c$.

III.5 <u>Acoustical Energetics</u>

The energy associated with a sound field in an ideal fluid is of two sorts: kinetic energy of motion and potential energy resulting from compression of the fluid. The strength of the sound field varies both in time and space, and we wish to determine the corresponding variations in energy. That is, <u>energy density functions</u> must be defined to describe the energy per unit volume in terms of the acoustic variables, pressure and particle velocity.

Throughout the development of this section we use real notation for the acoustic variables.

The kinetic energy of a mass m moving with instantaneous velocity v is $1/2mv^2$. Similarly the kinetic energy density of fluid with mass density ρ moving with instantaneous velocity u is

$$T = \frac{1}{2}\rho u^2 \quad . \tag{III.5.1}$$

The distinction between instantaneous density ρ in the presence of a sound wave and the average density ρ_o need not be maintained if the medium is stationary, since it involves only small terms of third order which we neglect.

The potential energy density equals the work, per unit volume, done in compressing the fluid. Consider a small volume element V_o which is compressed in the presence of sound to volumes

$$V = V_o[1-s(t)]$$

where the small quantity s, the condensation, fluctuates in time with vanishing average value (see Eq. III.2.1). The total instantaneous pressure is

$$P = P_o + p \quad ,$$

where P_o is the constant ambient pressure and p the sound pressure. Then the work done in compressing V_o from a value s=0 (i.e. ambient conditions) to a general value s is given by the integral

$$W = V_o \int_0^s P \, ds \quad .$$

This integral involves two terms. The first term, $P_o s$, vanishes in a time average; it is not, in any case, to be associated with the sound wave since it represents work done by the ambient "atmospheric" pressure P_o.* The second term is identified with the acoustic potential energy density:

$$U = W/V_o = \int_0^s p \, ds \quad . \qquad \text{(III.5.2a)}$$

A relation between pressure and condensation is required. It follows directly from the continuity equation (III.2.3a) and the equation of state (III.2.4b), whose combination yields

$$s = p/\rho_o c^2 \quad .$$

The integral is therefore very simple, and yields:

$$U = p^2/2\rho_o c^2 \quad . \qquad \text{(III.5.2b)}$$

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*Comparison with a similar question in mechanics will elucidate this neglect of the term involving P_o. Consider the dynamics of a mass-spring system (m,k) and its vibrations about the equilibrium position, x=0, which the mass assumes under a steady "external" force F_o. (Such a force, which may be gravitational in origin, is sometimes called a "body force".) If the potential energy is equated to work done on the spring in a displacement x, its value is found to be $U = F_o x + 1/2\ kx^2$. That part involving F_o will lead (through $\partial L/\partial x$ where L is the Lagrangian) to a term F_o in Lagrange's equation of motion. However, a compensating term F_o appears in Lagrange's equation as the generalized force; the two terms cancel. Therefore, the dynamics of such a system are correctly predicted if the steady force F_o is neglected both in the potential energy and in the generalized force; indeed, such is the usual procedure. A careful justification of the neglect of the energy term involving P_o in the acoustic case will be found in H. Goldstein's Classical Mechanics (Addison-Wesley Publishing Company, Reading, Mass., 1959) section 11-3.

The sum of acoustic kinetic and potential energy densities is called, simply, the acoustic energy density:

$$E = T + U = \tfrac{1}{2}(\rho_o u^2 + p^2/\rho_o c^2) \quad . \tag{III.5.3}$$

There is one other energetic quantity of interest, the intensity. Intensity is defined as the rate of flux of energy through a surface, per unit of surface area. The intensity is a directed quantity, i.e. a vector function, since the flux of energy is also directed. Consider a small area A oriented normal to the particle velocity vector $\underline{u}=\dot{\underline{\xi}}$ The flux of energy δE through A in a small interval of time δt equals the work done by the fluid on the one side of A upon the fluid on the other. Thus, we compute the work in the displacement $\delta\xi = u\delta t$ which occurs in the time interval δt:

$$\delta W = [(P_o + p)A]u\delta t \quad ,$$

so that the intensity is

$$(\delta W/\delta t)/A = (P_o + p)u \quad .$$

The first term is work done by the ambient pressure, and is not counted as acoustic intensity; in the absence of a steady wind, its average value vanishes. The acoustic intensity is identified as the second term, that is

$$\underline{I} = p\underline{u} \quad , \tag{III.5.4}$$

a vector with the direction of the particle velocity. (It is readily seen that there is no energy flux across a surface to which $\underline{u}$ is tangential. In general, the magnitude of the flux through a surface whose unit normal vector is $\underline{n}$ varies as the cosine of the angle between $\underline{u}$ and $\underline{n}$, i.e. as the scalar product $\underline{u}\cdot\underline{n}$.)

Let us apply these formulas to the case of a general plane wave travelling in the direction of increasing x coordinate. The acoustic pressure and particle velocity were found to be

$$p = F(x-ct)$$

$$u = p/\rho_o c \quad ,$$

where F can have any functional form, and the velocity vector has only an x-component (Eq. III.3.2). Then it is readily shown that

$$T \equiv \tfrac{1}{2}\rho_o u^2 = \tfrac{1}{2}E = \tfrac{1}{2}p^2/\rho_o c^2 \equiv U \quad ; \qquad \text{(III.5.5a)}$$

i.e., kinetic and potential energy densities are everywhere equal. Moreover the intensity in the x direction is

$$I \equiv pu = p^2/\rho_o c = cE \quad . \qquad \text{(III.5.5b)}$$

Note that I=cE is valid at every point and every instant. Now, this intensity was defined as the rate of flux of energy density. Therefore it is evident from this equation that the energy density of the wave is travelling in the direction of propagation with an <u>energy speed</u> equal to c, the "sound speed".

This equality must be considered a happy circumstance, peculiar to plane sound waves and a limited number of other cases. For, remember, the sound speed c was found to be the speed with which the pattern of pressure distortion propagates. That these two definitions are quite distinct is illustrated by the homely example of a long, crawling caterpillar. One observes in him a pattern of alternating distortion of the body which progresses forward at a speed much greater than his average speed. Fortunately for the caterpillar, the speed of energy flux is not the same as the distortion wave speed, else energy would accumulate unpleasantly in his head.

A more precise, scientific distinction will be delayed until it becomes necessary, in the consideration of bending waves in structures.

III.6 Standing Waves

The previous examples of sound waves have been <u>progressive</u> or <u>travelling</u> waves, that is, pressure disturbances that propagate continuously in a specific direction at the speed of sound. A typical example is the plane wave whether a pure tone wave or not.

In contrast, the pressure disturbances in a <u>standing</u> wave do not propagate; the dependence of the <u>real</u> sound variables upon position and time is separable; e.g., as

$$p(\underline{r},t) = R(\underline{r})\ T(t) \quad . \qquad \text{(III.6.1)}$$

This description of the standing wave, as a constant spatial pattern of response, $P(\underline{r})$, which everywhere fluctuates in the same way $T(t)$, accentuates a feature which can be recognized as an important property of "resonances" or "natural modes of vibration." Indeed, standing waves find their greatest application in the study of the acoustical resonances of rooms.

Let us construct some standing waves from the superposition of several pure-tone plane waves. First, add to a wave travelling in the direction of increasing coordinate x a second wave of equal amplitude travelling in the opposite direction. The expressions for their sound pressures are

$$\begin{aligned} p_1 &= A \cos\ (\omega t - kx) \\ p_2 &= A \cos\ (\omega t + kx) \end{aligned} \qquad \text{(III.6.2a)}$$

where the origins of t and x have, without loss of generality, been chosen so that each signal has its maximum value at $t=0$, $x=0$. By a standard trigonometric identity, the total pressure is found to be

$$p = p_1 + p_2 = 2A \cos \omega t \cos kx \quad , \qquad \text{(III.6.2b)}$$

which is a one-dimensional standing wave. The corresponding particle velocities are readily found by application of the force equation. Let $\underline{e}$ be the unit vector in the direction of increasing x. The particle velocity vectors are

$$\begin{aligned} \underline{u}_1 &= \underline{e}\ p_1/\rho_o c \\ u_2 &= -\ \underline{e}\ p_2/\rho_o c \\ u &= \underline{e}\ (2A/\rho_o c)\ \sin \omega t \sin kx \quad . \end{aligned} \qquad \text{(III.6.2c)}$$

There are many interesting features that can be demonstrated for this one dimensional standing wave. For example, there is never any particle velocity on the plane $x=0$ through the origin, nor on any of the parallel planes $x=n\lambda/2$, an integral number of half-wavelengths from the first. These are called <u>nodal</u> planes for the particle velocity. On the other hand, the sound pressure is a maximum on these planes, and vanishes on another set of planes located at the mid-points $x = (n+1/2)\lambda/2$ between the first set. Thus, there are nodal planes for pressure which are distinct from those for velocity.

It is well known that a plane travelling wave striking a rigid wall at normal incidence will be reflected as a wave of equal strength travelling in the opposite direction. If the rigid wall is located at, say, $x=0$, and a pure-tone plane wave approaches from negative values of x, the resulting steady-state sound field is indistinguishable in the region $x<0$ from the solution in Eq. III.6.2. A nodal plane for a velocity component is equivalent to a boundary condition of vanishing velocity, i.e., a rigid wall.

Let us introduce another rigid wall at another such nodal plane; for example at $x=-N\lambda/2$. Once the pure-tone sound wave has started travelling back and forth between the two walls, it will maintain exactly the standing-wave field given by Eq. III.6.2. (How the wave is started is an entirely different question.) Mathematically speaking, the standing wave is a solution to the one-dimensional Helmholtz equation ("steady-state wave" equation),

$$\frac{d^2p}{dx^2} + k^2p = 0 \quad ,$$

subject to rigid-wall boundary conditions

$$u = 0 \text{ , therefore } dp/dx = 0 \quad ,$$

at $x=0$ and $x=-N\lambda/2 = -N\pi/k$. While we are at it, we may connect the two planes by rigid walls parallel to the x-axis; the component of velocity normal to the x-axis vanishes everywhere. In this roundabout fashion, we have discovered a natural mode of a particular rigid-walled room. A more nearly complete study of natural modes will come later.

A sound field somewhat more complicated than the one-dimensional standing wave is generated by the superposition of two pure-tone plane waves of equal amplitude which travel in directions neither parallel nor anti-parallel. Let us use the complex convention and vector notation for their analysis. Suppose the wavenumber vectors of the two waves lie in the xy-plane and that each is the reflection of the other in the x-axis (see Fig. III.2A). The expressions for their pressures can then be written

$$p_1 = A \exp i(\omega t - \underline{k}_1 \cdot \underline{r}) = A \exp i(\omega t - k_x x - k_y y) \quad ,$$
$$p_2 = A \exp i(\omega t - k_x x + k_y y) \quad , \qquad \text{(III.6.3a)}$$

where the $\underline{k}$-vectors are

$$\underline{k}_1 = k_x \underline{e}_x + \underline{k}_y \underline{e}_y$$
$$\underline{k}_2 = k_x \underline{e}_x - k_y \underline{e}_y \quad ,$$

with the unit vectors in the x and y directions denoted by $\underline{e}_x$ and $\underline{e}_y$. The components of the $\underline{k}$-vectors must satisfy

$$k^2 = k_x^2 + k_y^2 \quad ,$$

and the angle θ that each makes with the x-axis is readily shown to satisy the relations

$$k_x = k \cos \theta \quad ,$$
$$k_y = k \sin \theta \quad .$$

The particle velocity vectors of the component waves are vectors with the directions of the $\underline{k}$-vectors:

$$\underline{u}_1 = (\underline{k}_1/k)(p_1/\rho_o c)$$
$$\underline{u}_2 = (\underline{k}_2/k)(p_2/\rho_o c) \quad . \qquad \text{(III.6.3b)}$$

(Compare Eq. III.4.8.)

Look now to the total pressure and the total particle velocity vector, which are the sums of the components. Still in complex notation, the total pressure is found to be*

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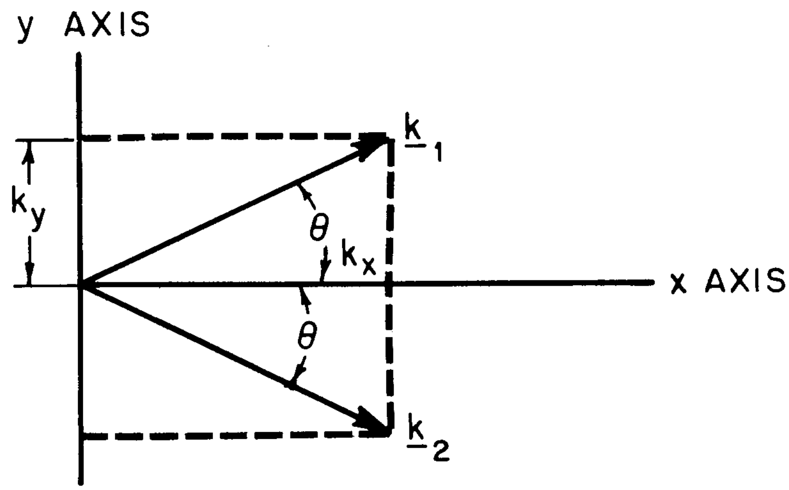
*The trigonometric identity essential for this section is

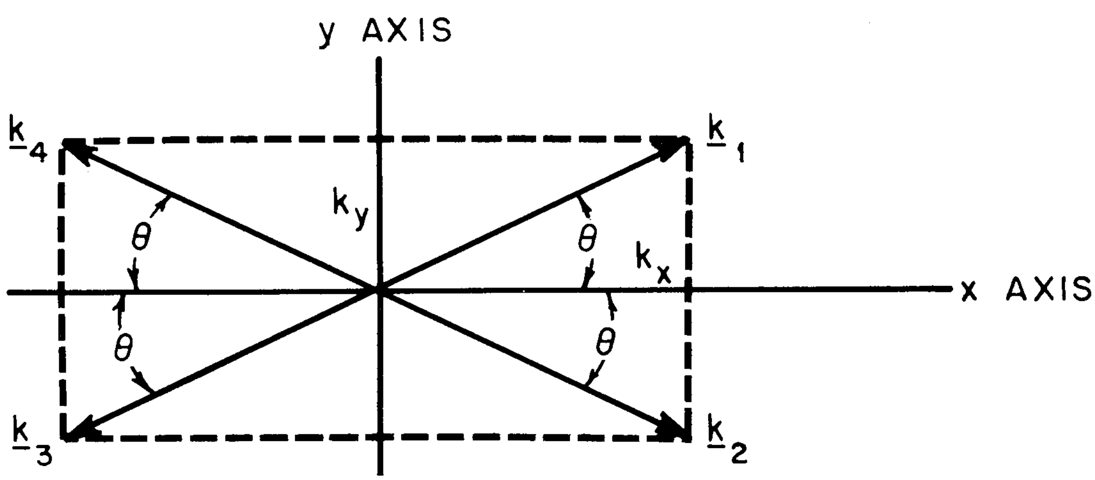
$$e^{iz} = \cos z + i \sin z \quad ,$$

whence follow

$$e^{iz} + e^{-iz} = 2 \cos z \ ,$$
$$e^{iz} - e^{-iz} = 2i \sin z \ .$$



(a) Sound pattern which is standing in y-direction and travelling in x-direction.



(b) Two-dimensional standing wave pattern.

Figure III.2.- Wavenumber vectors for plane wave components of various two-dimensional sound fields.

$$p = p_1 + p_2 = 2A \cos k_y y \exp[i(\omega t - k_x x)] \quad . \qquad \text{(III.6.4a)}$$

The expression for real pressure is

$$\mathrm{Re}\{p\} = 2A \cos k_y y \cos(\omega t - k_x x) \quad . \qquad \text{(III.6.4b)}$$

This sound field is a hybrid whose dependence on y is that of a standing wave, but which is travelling into the positive x direction. We note in passing that this sound field could have been generated by the reflection of the pure-tone plane wave p_1 from a rigid plane wall on the x-axis, y=0. (The y-component of total particle velocity is, by the force equation, proportional to $\partial p/\partial y$ and vanishes on those planes where $\sin k_y y=0$.)

A two-dimensional standing wave can be generated by four pure-tone plane waves of equal amplitude. Consider the four waves

$$p_j = A \exp i(\omega t - \underline{k}_j \cdot \underline{r}) \quad , \qquad j = 1,2,3,4,$$

whose $\underline{k}$ vectors are the four combinations of

$$\underline{k}_j = \pm k_x \underline{e}_x \pm k_y \underline{e}_y$$

with different pairs of signs (see Fig. III.2B). These four vectors are the successive reflections of any one in both the x and the y axes. They all lie in the same plane. The expression for their total sound pressure is readily found to be

$$\left.\begin{aligned} p &= 4A \cos k_y y \cos k_x x \, e^{i\omega t} \\ \mathrm{Re}\{p\} &= 4A \cos k_y y \cos k_x x \cos \omega t \quad . \end{aligned}\right\} \qquad \text{(III.6.5)}$$

In this two-dimensional standing wave the y-component of particle velocity, proportional to $\partial p/\partial y$, vanishes on the planes

$$\sin k_x y = 0, \quad k_y y = n_y \pi \quad , \qquad n_y \text{ any integer,}$$

and the x-component of velocity vanishes on the orthogonal planes

$$\sin k_x x = 0, \quad k_x x = n_x \pi \quad , \qquad n_x \text{ any integer.}$$

Moreover, the z-component of particle velocity vanishes everywhere. Just as in the case of the one-dimensional standing wave, a rigid boundary may be substituted for any one of these nodal planes. Indeed, a cell can be constructed from rigid walls at two of the planes with constant y and at two of the planes with constant x. Then this cell may be closed off by walls at any two planes of constant z to form a rectangular room. The sound field given by Eq. III.6.5 is a correct solution to the steady-state boundary value problem for the room.

There are several points to note about this two-dimensional wave. First, the standing wave is constructed from four plane waves of equal amplitude, whose $\underline{k}$-vectors all lie in a single plane and which include two arbitrary directions and their two opposites. Second, the resulting total sound field is periodic along both the x- and the y-axis; the periodicities are determined by the projections of the $\underline{k}$-vectors upon those axes. For example, the periodicity in x is a distance Λ_x such that

$$k_x \Lambda_x = 2\pi \quad .$$

The value of k_x is the projection of the $\underline{k}$-vector on the x-axis:

$$k_x = |\underline{k}_j \cdot \underline{e}_x| = k \cos\theta \quad ,$$

where $k = 2\pi/\lambda$ and θ is the angle between the $\underline{k}$-vector and the x-axis. Whereas a plane progressive wave has periodicity λ in the direction of propagation, the periodicity Λ_x is found to be

$$\Lambda_x = \lambda/\cos\theta \quad .$$

The quantity k_x is called a trace wavenumber, since it is the wavenumber associated with the "trace" of the sound field upon the x-axis. The quantity Λ_x is correspondingly called a trace wavelength. (See Fig. III.3.)

It is important to remember that the periodicity of a sound field along an axis inclined to its direction of propagation is determined, when appropriate, from a geometric projection of the $\underline{k}$-vector upon that axis. It is NOT determined by projecting the wavelength.

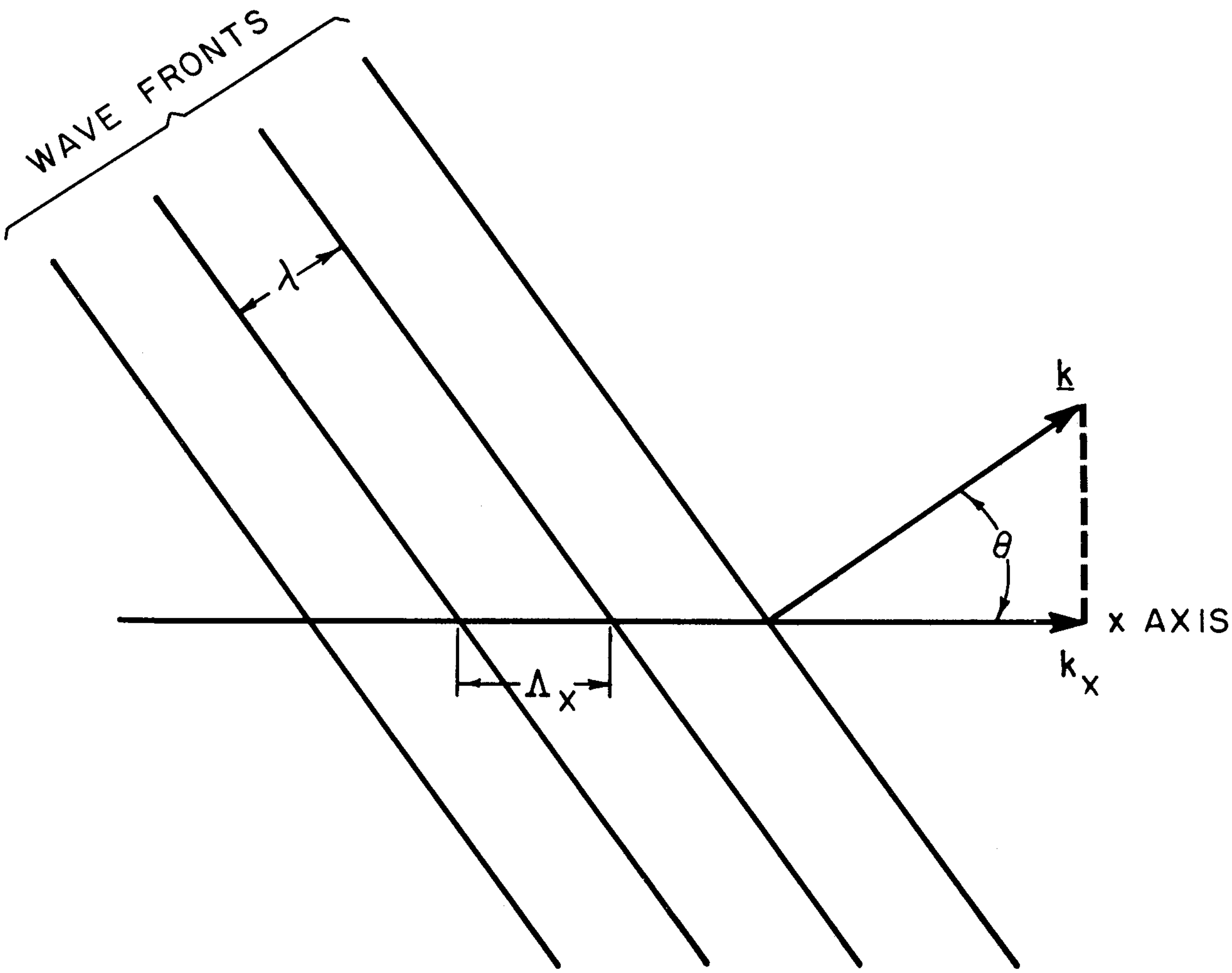


Figure III.3.- Plane wave; trace wavenumber k_x, trace wavelength Λ_x.

The concept of trace wavenumbers will play a central role in the analysis of the coupling between sound and structural vibration. The trace wavenumbers determine the spatial periodicities of the sound pressure which exists on a plane inclined to the direction of propagation of the sound wave. Thus, the spatial characteristics of the forces exerted by the sound wave upon a structure lying in this plane are determined by the trace wavelengths. The structural vibration excited by sound is strongly dependent upon the distribution of forces.

As a final example of standing waves, the analysis for a three-dimensional one will be sketched briefly. By a straightforward generalization of the preceding work, it is found that we require eight pure-tone plane waves of equal amplitude

$$p_j = A \exp i(\omega t - \underline{k}_j \cdot \underline{r}) \quad .$$

Their $\underline{k}$-vectors must have non-vanishing components on each of the three orthogonal coordinate axes, and can be generated by reflection in the three axes of any one vector, and by reflection of its reflections; they are the eight combinations with different signs of

$$\underline{k}_j = \pm k_x \underline{e}_x \pm k_y \underline{e}_y \pm k_z \underline{e}_z \quad .$$

The relations between the trace wavenumbers and the wavenumber $k=\omega/c$ involve the direction cosines of the $\underline{k}$-vector:

$$k_x = \left| \underline{k}_j \cdot \underline{e}_x \right| = k \left| \cos \theta_x \right| \quad ,$$

where θ_x is the angle between the $\underline{k}$-vector and the x-axis. Thus the trace wavenumbers satisfy the relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 \quad .$$

The total sound pressure is the standing wave

$$p = 8A \cos k_x x \cos k_y y \cos k_z z \, e^{i\omega t} \quad . \qquad \text{(III.6.6)}$$

This field has three sets of nodal planes on which the normal component of particle velocity vanishes, each set being perpendicular to one of the coordinate axes.

III.7 Room Acoustics (Part I)

The previous considerations have pertained, by and large, to the behavior of sound waves in unobstructed, "free" space. Boundaries to the fluid have been ignored, except insofar as they could be introduced post hoc, as rigid walls were introduced at the nodal planes of standing waves.

We must now turn to another extreme, sound fields in rooms, choosing as a model for careful study a perfectly rectangular room with hard walls.* The concepts developed for such a system will later be extended qualitatively to more realistic cases.

Consider a rectangular room with edges of length L_1, L_2, and L_3. In a cartesian coordinate system (x_1,x_2,x_3), the room can be oriented so that it lies in the ranges $0 \le x_1 \le L_1$, $0 \le x_2 \le L_2$, $0 \le x_3 \le L_3$. We wish to find the natural modes of oscillation; i.e. those special pure-tone sound fields which, once started, will ring steadily ever after. (We assume there is no dissipation.) These are the solutions to the acoustic wave equation for pure tones which satisfy the boundary conditions, zero value of the normal component of particle velocity at every wall.

Let the sound pressure associated with a typical natural mode, the M-th, be denoted by

$$P_M = A_M \psi_M(x_1,x_2,x_3) \exp(i\omega_M t) \qquad \text{(III.7.1a)}$$

where A_M is an arbitrary constant, ω_M is the natural frequency, and ψ_M is a function defining the pressure pattern in space. The sound pressure must satisfy the wave equation, which reduces in this pure-tone case to

$$\nabla^2\psi_M + k_M^2\psi_M = 0 \quad , \qquad k_M \equiv \omega_M/c \quad . \qquad \text{(III.7.1b)}$$

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*For a detailed review of this wave theory of room acoustics, see P. M. Morse and R. H. Bolt, "Sound in Rooms," Reviews of Mod. Phys., 16, No.2(April, 1944). A good summary is available in P. M. Morse, Vibration and Sound (McGraw-Hill Book Company, New York, 1948), Chapter VIII.

The boundary conditions require the vanishing of the gradient of pressure in a direction normal to each rigid wall. With the subscript notation of coordinates, the boundary conditions can be written concisely:

$$\partial\psi_M/\partial x_j = 0 \quad , \qquad x_j = 0 \text{ and } L_j; \qquad j = 1,2,3.$$

The wave equation and boundary conditions are both separable, and the solution can be obtained by the method of separation of variables. The typical modal solution to the wave equation (also satisfying the boundary conditions at $x_j=0$) is

$$\psi_M = \cos k_1x_1 \cos k_2x_2 \cos k_3x_3 \quad , \tag{III.7.2a}$$

subject to the relation

$$k_1^2 + k_2^2 + k_3^2 = k_M^2 \equiv \omega_M^2/c^2 \quad . \tag{III.7.2b}$$

The boundary conditions at $x_j=L_j$ furnish the addition restrictions

$$k_j = m_j\pi/L_j, \; m_j \text{ any integer}; \qquad j = 1,2,3. \tag{III.7.2c}$$

The last two relations are readily combined into the <u>frequency equation</u> for computing the natural frequency of the M-th mode:

$$\omega_M^2 = (\pi c)^2 \sum_{j=1}^{3} (m_j/L_j)^2 \quad . \tag{III.7.3}$$

It is apparent that all the identifying characteristics of a mode are determined by the three integers (m_1,m_2,m_3) and that the previously mysterious letter M is merely shorthand for the triad. Only non-negative values of the integers are required, since neither ψ_M nor ω_M can be distinguished by the signs of the m_j.

The frequency equation, giving ω_M^2 as a weighted sum of squares of integers, lends itself readily to a geometric interpretation. In a three-dimensional cartesian coordinate system, plot a point with coordinates $\omega_j \equiv m_j(\pi c/L_j)$ for particular values of the integers (Fig. III.4). Then, the distance from the origin to that point equals the natural frequency ω_M of that mode. The coordinate system in which this plotting is done is usually

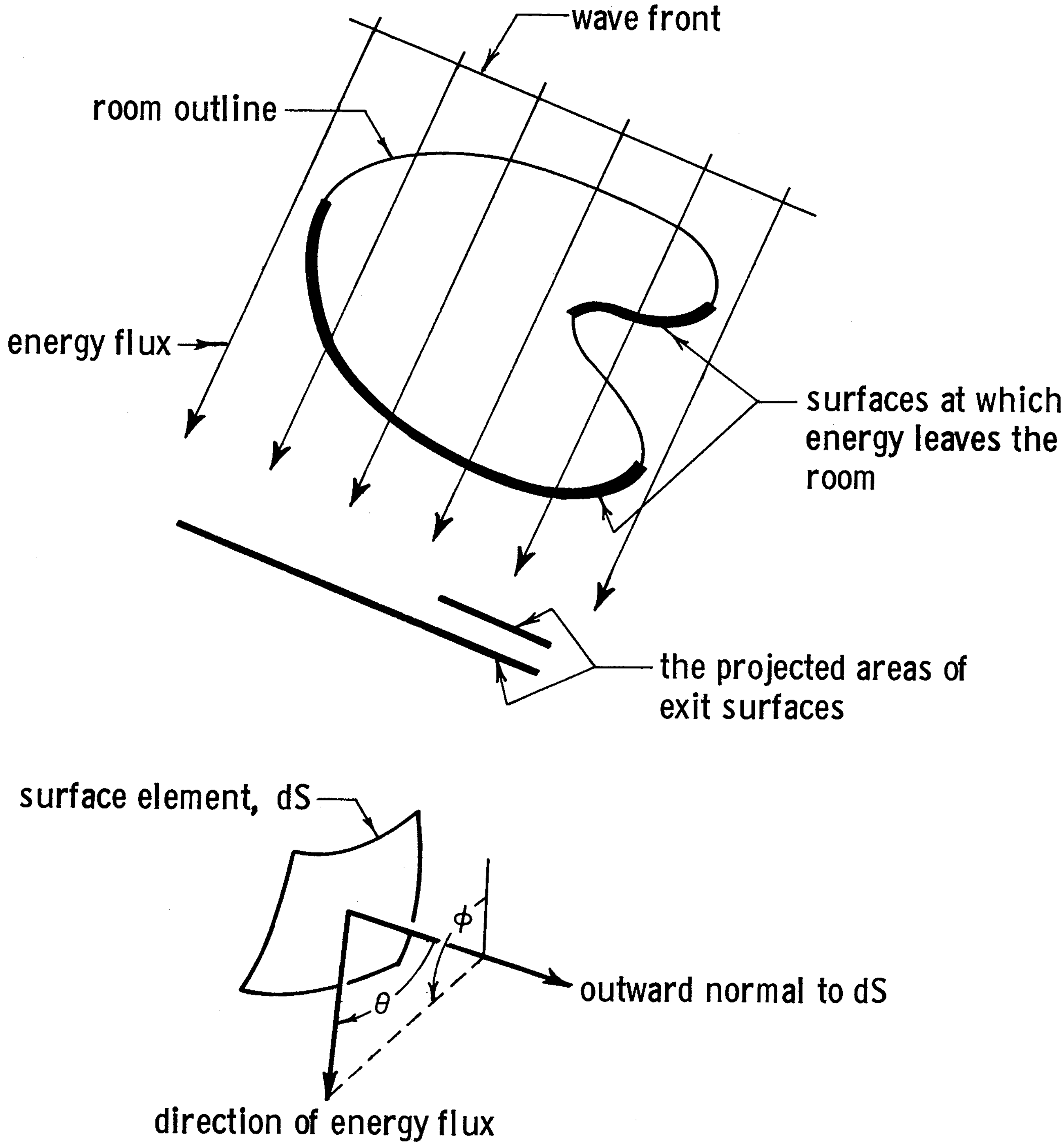


Figure III.4.- Geometrical constructions for evaluating mean free path.

called frequency space, although no particular physical importance need be attached to the separate coordinates ω_j of a point. The collection of points in frequency space for all the normal modes of a room form a uniform lattice completely filling one quadrant.

Modal Densities: The number of modes $N(\omega)$ having natural frequencies less than some value ω is readily found from this geometrical interpretation. Let us associate with each lattice point the rectangular volume element extending half-way to its nearest neighbors. That volume element is the same for each point and equals

$$\pi^3 c^3/V \quad ,$$

where $V \equiv (L_1 L_2 L_3)$ is just the volume of the room itself. We associate the same volume element with the points lying on the flat surfaces of the octant (corresponding to modes with just one m_j equal to zero) and with points on the octant's edges (corresponding to modes with two m_j equal to zero). Then, the total volume filled by the elements for points with $\omega_M < \omega$ is

$$N(\omega)\ \pi^3 c^3/V \quad ,$$

and essentially fills the octant out to a radius ω. (We say "essentially" since the outer surface of the volume filled by the elements is only a rectangular approximation to the smooth sphere.) For a first approximation, one may equate the two ways of computing the volume of a spherical octant:

$$\begin{aligned} N(\omega)\pi^3 c^3/V &\approx \pi\omega^3/6 \\ N(\omega) &\approx \omega^3 V/6\pi^2 c^3 \quad . \end{aligned} \tag{III.7.4}$$

Correction terms can also be computed for the extent to which the volume filled by the elements exceeds the octant, in a regular manner, on the latter's flat surfaces and along its edges. The corrections to Eq. III.7.4 are additive terms, found to be

$$(\omega^2 S/16\pi c^2) + (\omega P/16\pi c) \quad ,$$

where S is the total surface area and P the total length of edges in the room.* The true number of modes with natural

*See P. M. Morse, Vibration and Sound (McGraw-Hill Book Company, New York, 1948), section 32.

frequencies less than ω is, of course, a stepped, discontinuous function of frequency, to which this three-termed formula yields a continuous approximation.

The derivative with respect to frequency of the continuous function is a measure of the average rate of increase with frequency of the number of modes, and is called the modal density (in frequency)

$$n(\omega) \equiv dN/d\omega \approx \omega^2 V/2\pi^2 c^3 \quad \text{sec.} \tag{III.7.5}$$

The reciprocal of the modal density is a measure of the average frequency interval between modes, although it is not precisely the average of the intervals.

The lattice points representing modes in frequency space are subject to another interpretation. In the section on standing waves, we saw that sound fields of precisely the form of Eq. III.7.2 could be constructed by adding a number (2, 4, or 8) of plane waves of equal amplitude; the two descriptions, by travelling waves and by modes, are completely equivalent. Those plane-wave components have $\underline{k}$-vectors given by the various combinations of sign in the expression

$$\pm k_1\underline{e}_1 \pm k_2\underline{e}_2 \pm k_3\underline{e}_3$$

where the $\underline{e}_j$ are unit vectors along the three edges of the room. The directions of the $\underline{k}$-vectors are the directions of propagation of the component waves; all can be constructed from any one vector by multiple reflections in the coordinate axes and planes (i.e., by sign changes of the parts).

Consider one component wave of the travelling wave description of a particular natural mode, choosing that component with the $\underline{k}$-vector

$$\underline{k}_M = \sum_{j=1}^{3} (m_j\pi/L_j)\underline{e}_j \quad ,$$

where all coefficients are positive. When the method of constructing the lattice of modes in frequency space is reviewed, it is readily seen that $c\underline{k}_M$ is identical with a vector in frequency space extending from the origin to the lattice point for the M-th mode. In other words, except for the scale factor c (sound speed), the plot of modal points in frequency space is identical with the end-points of the $\underline{k}$-vectors of the travelling waves by which the modes can be described. The complete set of k-vectors for the travelling waves is generated by duplicating the frequency-space lattice, Fig. III.4, in the other seven octants.

Such a plot of the allowable k-vectors for sound in the room is called a plot of the modes in k-space. (In the present case, of sound waves, the plots in k-space and in frequency space are indistinguishable except for a uniform scale change; that will not be the situation when the wave speed is a function of frequency, as in dispersive media.) The plot of natural modes in k-space is an extremely important conceptual tool in the present method of analysis. It contains the essential modal information, not only on natural frequencies (implicitly, in proportion to distance from the origin) but also on the spatial variations of pressure (the magnitude and direction of the k-vectors, and the magnitude of the trace wavenumbers which are the geometric projections of the k-vector on various axes).

The significance of the correction terms in the expression for modal density can now be clarified. The term involving the room area S was found by counting those modes on the flat surfaces of the octant in frequency space. The k-vectors of these modes lie parallel to one of the walls of the room; the modes are two-dimensional standing waves. The correction term involving the room perimeter P was found by counting those modes on the edges of frequency space. Their k-vectors are parallel to one of the axes of the room; the modes are one-dimensional standing waves. It is evident from Eq. III.7.4 that the proportion of 1- and 2- dimensional modes in the total number becomes increasingly smaller as frequency increases. Thus, the important conclusion: except at low frequencies, the typical mode is a three-dimensional standing wave.

Distribution in Angle: We have, then, identified the direction in k-space from the origin to a modal lattice point as the direction of propagation of energy in the sound field of that mode. Let us consider how these modal directions are distributed. Suppose all the modes having natural frequencies in the range $\omega - 1/2\Delta\omega \leq \omega_M \leq \omega + 1/2\Delta\omega$ were excited at the same time. We ask how many modes would be detected by a directional microphone which responds only to energy travelling in a narrow beam of solid angles.

Denote by $\underline{\Omega}$ the unit vector in the direction the microphone points; let $\Delta\underline{\Omega}$ represent the solid angle (ster-radians) within which the microphone is sensitive. Finally, write the number of modes detected in the form

$$n'\Delta\omega\Delta\underline{\Omega} \quad .$$

We are interested in the dependence of n' upon frequency, ω, and direction, $\underline{\Omega}$. The calculation is readily carried out in $\underline{k}$-space, in much the same manner as the modal density in frequency was calculated.

The lattice points in $\underline{k}$-space are uniformly distributed at the centers of volume elements having volumes π^3/V where $V = L_1L_2L_3$ is the room volume. On the other hand, the total volume in $\underline{k}$-space corresponding to ranges $\Delta\omega$ and $\Delta\underline{\Omega}$ at a radial distance $k = \omega/c$, is

$$k^2\Delta\underline{\Omega}\Delta k = (\omega^2/c^3)\Delta\underline{\Omega}\Delta\omega \quad .$$

We equate this volume to the product of the number of detectable modes and the volume per modal point, and get an approximate value for n':

$$n'(\omega,\underline{\Omega}) \approx \omega^2 V/\pi^3 c^3 \qquad \text{(III.7.6)}$$

The conclusion to be drawn is that n', the "density" of modes in frequency and in solid angle, is independent of the direction for which it is evaluated. Of course, one must recognize that the number of modes detected for any particular values of the four parameters ($\omega,\Delta\omega,\underline{\Omega}$, and $\Delta\underline{\Omega}$) will fluctuate with changes in any one of them. However the density n' is an average value, which becomes more accurate as the number of modes detected is increased.

Note the simple relation between the two different density functions, Eqs. III.7.5 and III.7.6:

$$n'(\omega,\underline{\Omega}) = (2/\pi)n(\omega) \quad .$$

The difference is related to the different concepts that are involved. The density $n(\omega)$ counts each mode (i.e. standing wave) as a single unit. The density $n'(\omega,\underline{\Omega})$ counts each travelling wave as a single unit. A typical mode is made up of eight travelling waves. The travelling waves corresponding to all modes are distributed uniformly in all directions, i.e. in 4π ster-radians; thus is explained the factor $2/\pi = 8/4\pi$.

Energetics of Sound in Rooms: The usual measuring instrument for determining the "strength" of sound indicates the time-average of the square of sound pressure, or the square root of that quantity: the rms (root mean square) pressure. The pressure is of particular interest to us as the exciting force for structural vibration. Here we consider relations between the sound pressure and the energy of the sound field in a room. Particularly, we relate the space-time average of squared-pressure to the space-time average of energy density, that is, the time average of total energy divided by the room volume.

The analysis to follow refers to the modal pressure distributions found in the particular case of a rectangular room. But it should be noted that those particulars are not at all essential to the derivation of the average relations which are based solely on general modal concepts and the general relation between acoustic potential energy density and the square of sound pressure.

The pressure response in a single mode of a rectangular room can be written (Eqs. III.7.1 and III.7.2)

$$p_M(\underline{r},t) = P_M \psi_M(\underline{r}) \cos \omega_m t \qquad \text{(III.7.7a)}$$

with

$$\psi_M = \sqrt{\epsilon_M} \cos k_1 x_1 \cos k_2 x_2 \cos k_3 x_3 \quad . \qquad \text{(III.7.7b)}$$

The factor $\sqrt{\epsilon_M}$ is an arbitrary normalizing factor whose value we choose to satisfy the condition that the average value of ψ_M^2 shall be unity:

$$\langle \psi_M^2 \rangle_{\underline{r}} = \frac{1}{V} \int \psi_M^2 \, dx_1 dx_2 dx_3 = 1 \quad . \qquad \text{(III.7.7c)}$$

Since the allowable values of k_j for natural modes are restricted by the condition $k_j L_j = m_j \pi$ with m_j an integer, the room contains integral numbers of half-periods of the periodic functions $\cos k_j x_j$. It follows that the average value of $\cos^2 k_j x_j$ over the range 0 to L_j is 1/2; the normalizing factor is thus found to be

$$\epsilon_M = \begin{Bmatrix} 8 \text{ , all } k_j \neq 0 \\ 4 \text{ , only one } k_j \\ 2 \text{ , two } k_j = 0 \end{Bmatrix} = 0 \quad . \qquad \text{(III.7.7d)}$$

The average in both space and time of the square of pressure is now readily calculated:

$$\langle p_M^2 \rangle_{\underline{r},t} = \frac{1}{2} P_M^2 \quad .$$

Now, the instantaneous potential energy density is everywhere proportional to square of pressure (Eq. III.5.2); therefore, the space-time averages are also proportional:

$$\langle U_M \rangle_{\underline{r},t} = (1/2\rho_o c^2) \langle p_M^2 \rangle_{\underline{r},t} \quad . \qquad \text{(III.7.8a)}$$

The kinetic energy could be evaluated from its definition, once the particle velocity is found, but that analysis would be an inelegant substitute for well-known physical principles. In a physical system resonating naturally without external forces, the time averages of total kinetic and potential energies are equal (cf. Hamilton's Principle); the total energy is constant in time. The corresponding relations for the energy density functions and their space and time averages are:

$$\langle U_M \rangle_{\underline{r},t} = \langle T_M \rangle_{\underline{r},t} \quad ,$$

$$\langle E_M \rangle_{\underline{r}} = \langle E_M \rangle_{\underline{r},t} \equiv \langle U_M + T_M \rangle_{\underline{r},t} = 2 \langle U_M \rangle_{\underline{r},t} \quad .$$

The space average energy at every time is therefore simply related to the space-time average of p^2:

$$\langle E_M \rangle_{\underline{r}} = \langle p_M^2 \rangle_{\underline{r},t} / \rho_o c^2 \quad . \qquad \text{(III.7.8b)}$$

These relations have pertained to a single mode. Suppose the pressure in the room is due to many modes. It is a well known mathematical principle that the solutions for different natural modes are spatially orthogonal, that is,

$$\langle \psi_M \psi_N \rangle_{\underline{r}} = 0 \quad , \qquad M \neq N \quad .$$

It follows that the average squared-pressure is a sum of modal contributions without cross-products:

$$\langle (\Sigma p_M)^2 \rangle_{\underline{r}} = \Sigma \langle p_M^2 \rangle_{\underline{r}} \quad ,$$

and that the modal energies are also simply additive. In other words, the relations in Eqs. III.7.8 between average energy densities and average squared-pressure are equally as valid for a sum of modes as for a single one. Moreover, as was noted in the introduction, the relations are valid for a room of any shape.

Mean Free Path: The general sound field in a room can be expressed as a superposition of responses in the various natural modes. We have seen in the case of the rectangular room how each mode, and therefore the general sound field, is also strictly describable as a superposition of travelling plane waves.

For later purposes, we wish to evaluate the time-rate at which sound energy is incident upon the room walls. In a real room, the walls are not perfectly rigid and they do not perfectly reflect the incident sound; a small fraction of the incident energy is absorbed instead of reflected. The amplitude of sound left to its own devices will gradually decay. If the rate of decay is small, when compared with the period of oscillations, the modal description of the sound field developed for the room without energy losses can be retained as a good approximation to the spatial distribution of sound pressure and energy. The gradual decay is incorporated as a gradual change in the modal amplitude, i.e. in the modal energy. The decay rate is evaluated by equating the rate of change of modal energy to the power dissipated by absorption. (This sort of approach to the slightly dissipative resonance is common to all fields of physics: electronics, mechanics, etc.) The sound absorptive properties of walls are generally characterized by "energy absorption coefficients", i.e. the fraction of the energy of an incident wave which is absorbed. Hence, our present interest in the rate of incidence of energy -- the incident power.

Suppose that all the modes of a rectangular room having natural frequencies in some relatively broad band are equally excited. The sound field in the room is described as a superposition of travelling plane waves (defined for all space), each having the same intensity but travelling in different directions. Consider one wave, travelling in a direction $\underline{\Omega}$, whose energy density is E and intensity is I=cE, (Eq. III.5.5), where c is the energy speed which equals the sound speed. Then the sound power incident upon the wall surfaces is just the product

$$\Pi(\Omega) = I \; P(\underline{\Omega}) \qquad ,$$

where $P(\underline{\Omega})$ is the "projected area" of the room, i.e. the area of its optical shadow for light travelling in the direction $\underline{\Omega}$. (See Fig. III.5 which illustrates the more general case of a non-rectangular room with re-entrant surfaces; then, some parts of the optical shadow must be counted twice.)

This power pertains to a single direction of travel. We shall average it with respect to direction on the assumption that the incident intensity I is the same for all directions. A sound field satisfying this assumption is known as a <u>diffuse field</u>; sound fields in large, irregular rooms are found experimentally to be nearly diffuse. As we saw in connection with Eq. III.7.6, the directions associated with the component plane waves of modes in a frequency band $\Delta\omega$ are uniformly distributed in angle. Therefore the assumption that I is independent of direction corresponds to an assumption that all the wave components in some frequency band $\Delta\omega$ are excited to equal amplitudes. We desire the average value

$$\langle \Pi(\underline{\Omega}) \rangle = I \langle P(\underline{\Omega}) \rangle \quad ,$$

and must therefore average the projected area. This averaging is most easily performed first for a small element of the wall, later integrating over all wall elements.

Consider an element of wall area dS in Fig. III.5. We describe directions of incidence by the local spherical angle coordinates, θ and ϕ, arranged so that $\theta=0$ is the outward normal to dS, pointing <u>away</u> from the room. The differential of solid angle is

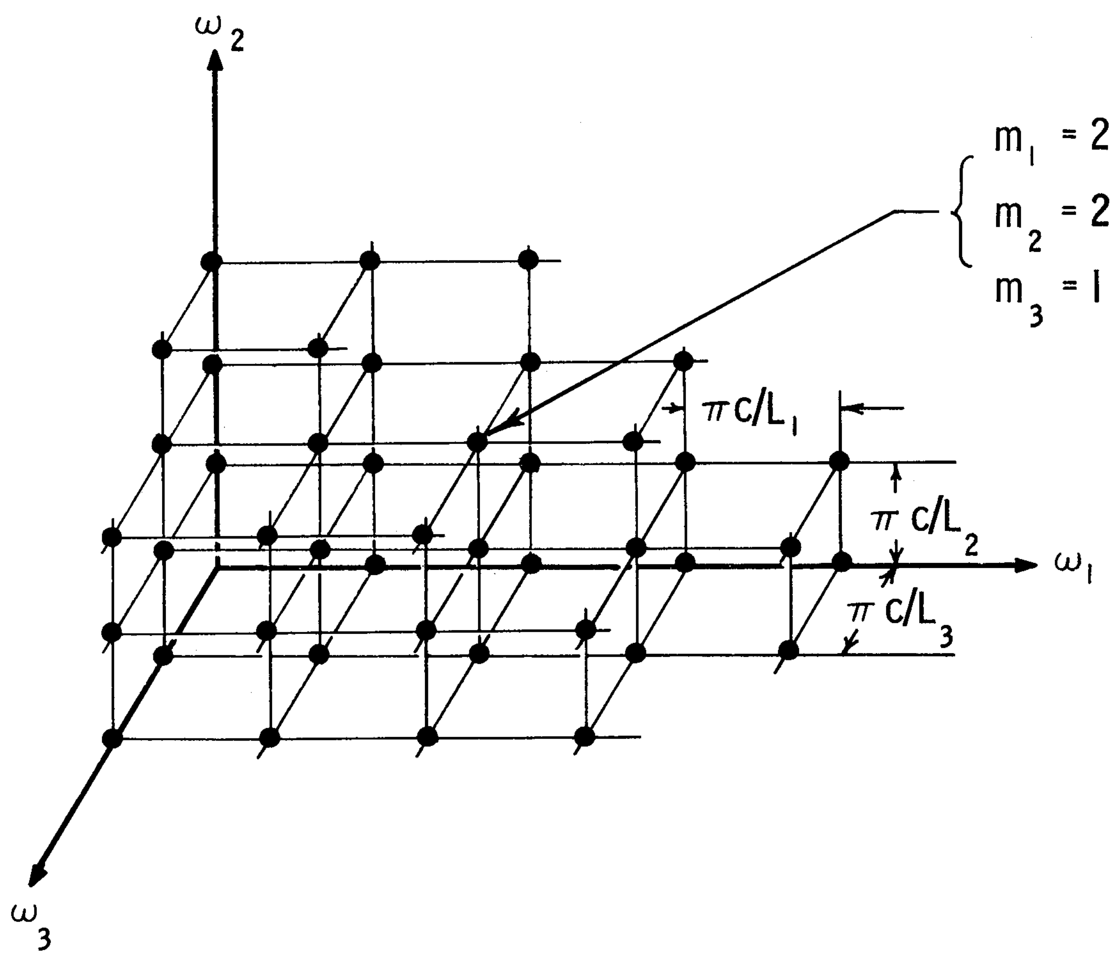
$$d\Omega = \sin\theta \, d\phi \, d\theta$$

and the total solid angle for all directions is

$$\int d\Omega = 4\pi \quad .$$

The projected area for given directions is $dS|\cos\theta|$, so that the average for all directions is

$$\frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} dS|\cos\theta| \sin\theta d\phi d\theta = dS/4 \quad .$$



$$\omega^2_{m_1, m_2, m_3} = \omega_1^2 + \omega_2^2 + \omega_3^2$$

$$\omega_j = m_j\,(\pi c/L_j), \; j = 1, 2, 3$$

Figure III.5.- Natural modes plotted as points in frequency space, for rectangular room. Distance from origin to lattice point of a mode equals natural frequency.

The integral over θ extends only from 0 to $\pi/2$, because, by definition, $P(\Omega)$ includes only those surfaces for which energy is incident upon the wall from inside the room. Putting these formulas together, we find that the average sound power incident on the walls is

$$\langle \Pi \rangle = I \langle P \rangle = I \int \frac{dS}{4} = IS/4 \qquad \text{(III.7.9a)}$$

where S is the <u>total</u> wall area of the room.

The total sound energy in the room is also proportional to I. For each angle of incidence, the energy in the room is

$$VE(\underline{\Omega}) = VI/c = \langle VE(\underline{\Omega}) \rangle \quad , \qquad \text{(III.7.9b)}$$

where V is room volume and E is the energy density of the wave. The result is a constant for all directions, and is therefore the same in the average.

The ratio of power incident upon walls to the total energy in the room assumes a very simple form:

$$\nu \equiv \frac{\langle \Pi \rangle}{\langle VE \rangle} = \frac{cS}{4V} \quad . \qquad \text{(III.7.10)}$$

This formula affords a simple way of calculating the power incident upon the walls from the energy, or from the space-time mean square pressure using Eq. III.7.8b. It is a fundamental formula of room acoustics. Constant use in experimental problems has validated its pertinence to practical noise fields, even though the diffuse field assumption on which it is based appears at first to be a gross idealization.

This ratio ν is called the <u>mean collision rate</u>, a name which unfortunately refers to a quite different method of calculating the same number. The length defined by

$$d \equiv c/\nu = 4V/S \qquad \text{(III.7.11)}$$

is called the <u>mean free path</u>. These apparently irrelevant names arose naturally in the solution of the following problem. Consider a pellet moving at speed c along straight paths within the confines of the room. Upon collision with a wall, the pellet takes a new straight path in some other direction. Assume that the totality of paths are so distributed that the density (paths per unit area) of paths perpendicular to any plane is a constant, both as to position on the plane and as to orientation of the plane. Subject to that assumption, d is the average length of all the paths and ν is the average time-rate of collisions of the pellet with walls. The essential identity of the mathematics of the two problems is readily seen.

Interestingly enough, analysis indicates that a single pellet in an ideally perfect rectangular room, being reflected by the walls as is light by a mirror, will not generally take up the required distribution of paths no matter how many paths nor even how many initial directions are included. (Differences up to 18% in the mean free path have been calculated*.) Of course, one may well question the pertinence of such pellet paths to the behavior of sound in real rooms.

One must recognize that the validity of many results of the present analysis is based on the plane wave description of modes in a rectangular room, and has not been demonstrated for other room shapes. We face one of the most difficult questions of room acoustics: how to make an analytical transition from simple room shapes to complex ones. The problem is largely unresolved. Fortunately some progress has been made. By exact analyses of unduly simplified rooms, by unduly simplified theories for real rooms, and by comparisons of theory and experiment, those situations which are susceptible to cautious predictions from various simple formulas are fairly well known.

For example, the description of a sound field by a superposition of orthogonal natural modes has wide validity. Therefore the relations between space-time averages of energy densities and of squared-pressure (Eqs. III.7.8) hold for the general shape. Moreover, it can be shown that the cumulative distribution and density of the natural frequencies for a room of any shape are correctly given by Eqs. III.7.4 and III.7.5, in the limit of high frequencies. (Only the first term, involving room volume, is important in the limit.)

In other respects, the present results for rectangular rooms cannot be generalized. However, we shall see that the hypothesis of uniformly distributed intensity flux (diffuse field) leads to experimentally validated results if the room is sufficiently irregular and non-ideal.

*F. V. Hunt, J. Acoust. Soc. Am. 36, 556-564 (1964).

III.8 Generation of Plane and Spherical Sound Waves

In this section we shall study some simple examples of sound sources which generate elementary wave types. We shall find that certain features of the methods we use and results obtained will carry over into our subsequent discussions on structural vibration and sound radiation.*

Sound Radiated by an Infinite Rigid Plane

Consider a large rigid plate that oscillates with a periodic velocity $v(t)$ in the direction of its normal. Let x be the coordinate normal to the plane. Then, the symmetry of the problem suggests that a plane wave is generated and propagates away in the + x direction. Therefore, the pressure in this wave must have the form

$$p(x,t) = F(x-ct) \quad ,$$

and, from Eq. III.3.2b, the particle velocity in the + x direction is

$$u(x,t) = F/\rho_o c \quad ,$$

where the functional form of F is still to be determined.

At the plate, $x = o$, we impose the boundary condition that the velocity of plate motion equal the particle velocity. This is equivalent to requiring that the fluid stay in contact with the plate. (This doesn't always happen; in liquids, the fluid sometimes breaks away from the vibrator giving rise to the phenomenon of cavitation.) Thus, the function F must satisfy

$$v(t) = u(0,t) = F(-ct)/\rho_o c = p(0,t)/\rho_o c \quad .$$

The pressure at $x = 0$ produces a reaction force per unit area on the plate. On an area A_p, the ratio of force F to velocity v of the plate is called the radiation impedance,

$$Z_{rad} = \frac{F}{v} = \frac{pA_p}{v} = \rho_o c A_p \quad . \qquad \text{(III.8.1)}$$

*These and many other examples are discussed in L. L. Beranek, Acoustics (McGraw-Hill Book Company, New York, 1954) Chapters 2, 4, and 5.

(Compare the use of impedance for a simple resonator in Chapter II, Section 6.) The power input to the plate required to generate this motion is

$$\Pi_{rad} = \langle Fv\rangle_t = \langle v^2\rangle_t \rho_o c A_p = \langle p^2\rangle_t A_p/\rho_o c = \langle v^2\rangle_t \; R_{rad} \qquad (III.8.2)$$

where $R_{rad} = \mathrm{Re}(Z_{rad})$. This formula closely approximates the sound power radiated by a finite flat plate of area A_p, when its dimensions are several acoustic wavelengths in extent. This power will also form a basis for comparison of other radiators; the ratio of the power radiated to that for a large flat plate of equal area is called the radiation efficiency and is denoted by σ_{rad}.

There are two significant features of Z_{rad} for the infinite flat plate. The first is that it is a real number, which means that for sinusoidal excitation, the force and velocity are in phase with each other. The second is that its magnitude is independent of frequency. Neither of these is true for the spherical source to be discussed next.

Sound Radiated by a Spherical Source

Next we consider the sound field generated by the uniformly pulsating small spherical source. The symmetry of the problem suggests that we employ spherical coordinates. Therefore, we must write the wave equation (Eq. III.2.6) in terms of spherical coordinates (r,θ,ϕ), which means we require the Laplacian operator ∇^2 in these coordinates:*

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 p}{\partial \phi^2} \quad .$$

Since there is no polar or azimuthal dependence in our source, we shall assume there is none in p, and drop derivatives in $\partial/\partial\theta$ and $\partial/\partial\phi$. We also introduce the variable $g = rp$. Doing this, one finds

$$\frac{\partial^2 g}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = 0 \quad , \qquad (III.8.3)$$

*F. B. Hildebrand, Advanced Calculus for Applications (Prentice Hall Inc., Englewood Hills, New Jersey, 1962), p. 305.

with a general solution

$$g = rp = H(r-ct) + K(r+ct) \qquad \text{(III.8.4)}$$

where H and K are arbitrary functions. The first term is a wave propagating in the +r direction (outgoing) and the second represents a wave travelling in the -r direction (incoming). (Compare section 3, above.)

The wave generated by our pulsating sphere will be outgoing; if there are no other sources or reflectors, this is the only wave. The pressure therefore is

$$p(r,t) = \frac{1}{r} H(r-ct) \quad .$$

In particular, if the time dependence is sinusoidal, then the pressure wave may be represented in complex notation by

$$p(r,t) = \frac{B}{r} e^{i(\omega t-kr)} \quad . \qquad \text{(III.8.5)}$$

The velocity is found as before by using the force equation III.4.2, and the gradient operator in spherical polar coordinates.* Neglecting θ and ϕ dependence as before,

$$\text{grad } p = \underline{e}_r \frac{\partial p}{\partial r} = - i\omega\rho_o \underline{u} \quad ,$$

where $\underline{e}_r$ is a unit vector in the direction of the radius vector. Unlike the velocity in the plane wave which is everywhere parallel, the velocity in this wave is everywhere directed radially outward. Combining these equations, one finds

$$u = \frac{p}{\rho_o c} \left(1 - \frac{i}{kr}\right) \quad . \qquad \text{(III.8.6)}$$

This is the same as the plane wave relation when kr>>1, but differs when kr is small.

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*F. B. Hildebrand, Advanced Calculus for Applications (Prentice Hall Inc., Englewood Hills, New Jersey, 1962), p. 305.

At the surface of the source $(r=a)$ $u = v$ and $F = p(a)A_s$, where F is the reaction force of the medium on the source and $A_s = 4\pi a^2$ is the source area. The radiation impedance, defined as the ratio of force to velocity on the surface, is

$$Z_{rad} = \frac{p(a)A_s}{v} = 4\pi a^2 \rho_o c \left(1 - \frac{i}{ka}\right)^{-1}$$

$$= 4\pi a^2 \rho_o c \frac{(ka)^2 + i(ka)}{1+(ka)^2} \qquad . \qquad \text{(III.8.7)}$$

Note that the impedance is complex and varies with frequency. The real part R_{rad} can be expressed as the product of a radiation efficiency and the radiation resistance for a plane wave:

$$R_{rad} = (\rho_o c \; A_s)\sigma_{rad} \qquad ,$$

whence we find

$$\sigma_{rad} = \frac{(ka)^2}{1+(ka)^2} \qquad . \qquad \text{(III.8.8)}$$

The radiation efficiency for the spherical source is plotted logarithmically in Fig. III.6.

The nature of Z_{rad} can perhaps be clarified by reference to Fig. III.7. Here we show a mass-dashpot system excited by a velocity v_s. Since the ideal dashpot has no mass, there is no difference in the magnitude of the force F at its ends, so that

$$F = R(v_s - v) = M \frac{dv}{dt} \qquad ,$$

or,

$$\frac{1}{R} F = v_s - \frac{1}{M} \int F \, dt \qquad .$$

If we differentiate this, we get

$$F + \frac{M}{R} \frac{dF}{dt} = M \frac{dv_s}{dt} \qquad .$$

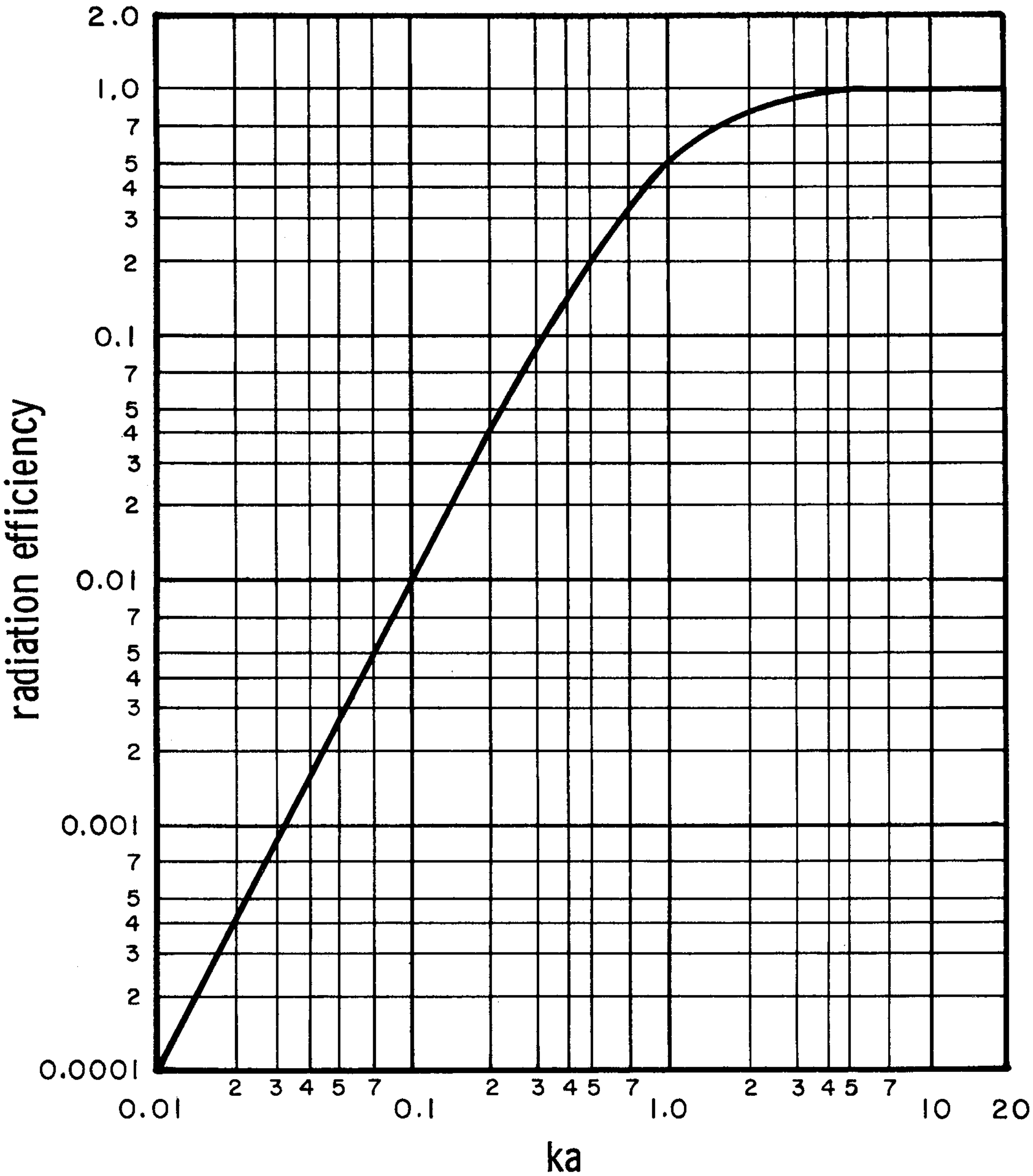

Figure III.6.- Radiation efficiency of a pulsating sphere of radius a.

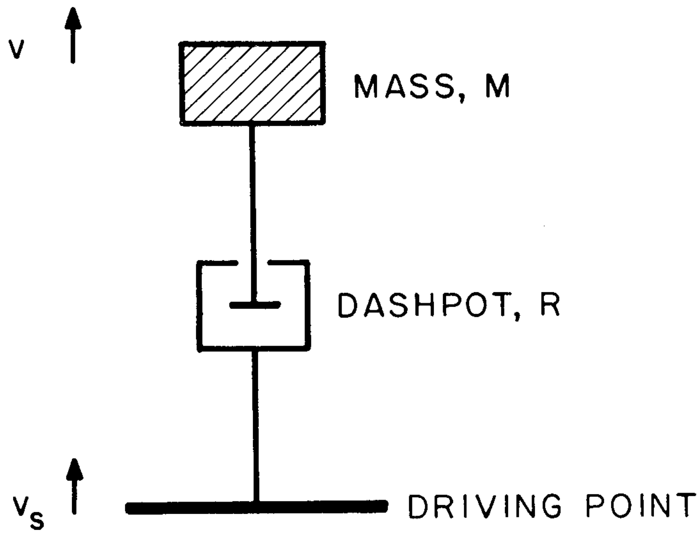

Figure III.7.- Mechanical equivalent of the radiation impedance for a simple source (pulsating sphere). (The mass equals three times the mass of fluid displaced by the sphere.)

If the velocity source is harmonic with frequency ω, we can write this equation in complex notation:

$$F\left(1+i\omega\frac{M}{R}\right) = i\omega M v_s \quad .$$

The ratio of force to velocity (input impedance for the mechanical system) is found to be

$$Z = \frac{F}{v_s} = R\,\frac{i\omega M/R}{1+i\omega M/R} \quad .$$

If we let $R = \rho_o c A_s$ and $\omega M/R = ka$, we get an expression identical with Eq. III.8.7; Fig. III.8 is, then, the mechanical equivalent circuit for the radiation impedance. Thus, the effective mechanical resistance is the high frequency limit $\rho_o c A_s$ and the effective mass is $M = \rho_o a A_s$. The acoustic medium presents a resistive (dissipative) and mass load to the pulsation of the sphere, with the mass load becoming less significant as the wavelength gets near or smaller than the radius of the source.

Radiated Power

The radiation resistance can be used to compute the radiated power (compare Eq. II.6.8):

$$\Pi_{rad} = \frac{1}{2} R_{rad}\left|v\right|^2 = \frac{\rho_o c A_s}{2}\left|v\right|^2 \sigma_{rad} \quad . \qquad \text{(III.8.9)}$$

We can also compute the radiated power by considering the sound field at some large distance from the source. No energy is dissipated in sound propagation, at least in the idealized case we consider. Therefore the power leaving the source should also pass through every larger surface enclosing the source. In particular, let us consider a very large spherical surface of radius r and area $4\pi r^2$. If r is large enough, the sound field there is, locally, approximately a plane wave; the intensity (power per unit area) can be found from the plane wave relationship, Eq. III.8.2. The intensity is the same everywhere on the sphere, since the sound field is spherically symmetric. Thus, we compute

$$\Pi_{rad} = 4\pi r^2 \left|p(r)\right|^2/2\,\rho_o c \quad , \qquad \text{(III.8.10)}$$

using $1/2|p|^2 = \langle[\mathrm{Re}(p)]^2\rangle_t$ in translating Eq. III.8.2. From Eq. III.8.5, we find

$$r\,p(r) = a\,p(a) \quad ;$$

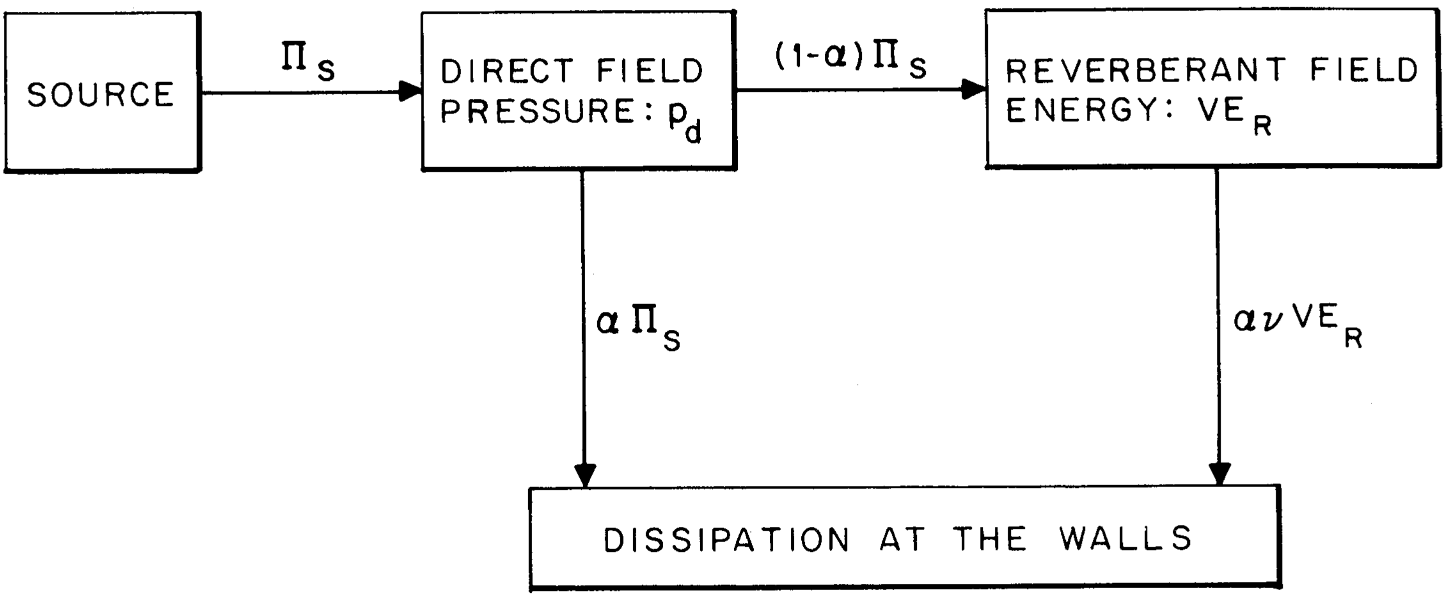

Figure III.8.- Power balance and energy diagram for reverberant rooms.

from Eq. III.8.7, we can express p(a) in terms of the velocity of the source, v. A little calculation confirms that the expression computed for power is the same as Eq. III.8.9.

III.9 Room Acoustics (Part II)

In the previous discussion of room acoustics, nothing was said of sound generation or dissipation. We considered only the ideal case, studying the undamped, natural response. Here we consider the practical case, introducing both a source and some sinks for the sound energy. In doing so, we use the approximate theory of room acoustics which is based on simple considerations of energy and intensity.*

Simple (Spherical) Sound Source in Room

We consider a small ($ka \ll 1$) spherical source which emits sound in a room. If the source is a velocity source (by which we mean that its surface normal velocity is prescribed independently of the loading pressure), then there is a fairly direct way of proceeding with the analysis. This is by finding the normal modes of the acoustic space with all its boundary conditions, including a boundary condition of vanishing normal velocity on the surface of the spherical source. The pressure field is then expanded in the eigenfunctions of the room and the response is computed. When the excitation is a band of noise instead of a pure tone, then the response must be integrated over the excitation spectrum and averaged in time.

The process just described is a very complicated one and, in fact, is unsolvable by purely analytical methods unless the geometry and the boundary conditions are highly idealized. An alternate approach, that of so-called ray or geometrical acoustics, has been developed which will predict many of the important features of the sound field in the situation described above. It has been so successful, as judged from its support by experimental analysis, that most calculations of sound in closed spaces are carried out in this way. The approach also has another importance for us: in many ways, it foreshadows the methods we shall apply to structures.

*The approximate theory and its use are explained in greater detail in L. L. Beranek, Acoustics (McGraw-Hill Book Company, New York, 1954) Chapter 10.

We assume that the source emits a spherical sound wave which progresses outward from the source until it encounters a wall. This is the so-called direct field of the source and the mean square pressure in it is related to the acoustic power of the source Π_s by Eq. III.8.10:

$$\langle p_d^2 \rangle_t = \frac{\rho_o c \, \Pi_s}{4\pi r^2} \quad , \tag{III.9.1}$$

where the subscript d stands for "direct field". (Note that we are back in real notation. Throughout this section, p stands for the real, physical pressure.)

Upon encountering a wall, the direct sound is partly absorbed and partly reflected. For some experimental or special purposes, one may attempt to have most of the energy absorbed at the first encounter with the wall. Such rooms are "anechoic", and may be used to simulate an outdoor situation. Other rooms are designed to have very hard, non-absorptive walls; they are called "reverberation chambers". In these, the sound reverberates for a relatively long time and the sound pressure from the reflections at the walls will be larger than the direct field from the source. The rooms one normally encounters in homes, offices, and schools are fairly reverberant.

The reflected sound will have a distribution of directions depending on the details of the scattering surfaces. One assumes that this distribution is diffuse, that is, completely uniform in angle. One also assumes that the resulting acoustic energy density E_R is uniformly distributed over the interior of the room. (The subscript R stands for reverberant.) Calculations based on these assumptions give surprisingly good answers for many rooms of practical interest.

Our description of the total sound field is now as follows. The source injects energy into the direct field; the energy travels to the room walls where a fraction of it is absorbed and the rest is reflected. The reflected energy makes up what is called the reverberant field, which is, we assume, a diffuse sound field with waves travelling in all directions. The reverberant field also loses energy at the room walls; a fraction of the incident power is absorbed.

This situation is very easily described analytically. First we define an energy absorption coefficient for the walls as the ratio of absorbed power to incident power:

$$\alpha \equiv \Pi_{absorbed}/\Pi_{incident} \quad . \tag{III.9.2}$$

We assume that the same coefficient applies to both the direct field and the reverberant field; α represents an average coefficient for all the walls. The power $\alpha\Pi_s$ is absorbed from the direct field, and the rest $(1-\alpha)\Pi_s$ is fed into the reverberant field to generate the energy density E_R and the total energy VE_R. (V is room volume.) The reverberant power incident on the walls can be computed from the mean collision rate ν (Eq. III.7.1); a fraction α of it is dissipated at the walls. The rate at which the reverberant energy changes is given by the power-balance equation

$$\frac{d}{dt}(VE_R) = (1-\alpha)\Pi_s - \alpha\nu(VE_R) \quad . \tag{III.9.3}$$

This power balance is indicated schematically in Fig. III.8.

In the steady state, the energy is constant, so that (with Eq. III.7.1) the reverberant energy density is

$$E_R = \frac{\Pi_s}{\nu V}\,\frac{1-\alpha}{\alpha} = \frac{\Pi_s}{c}\,\frac{4}{R} \tag{III.9.4}$$

where $R \equiv S\alpha/(1-\alpha)$ is called the room constant. (S is the total wall area of the room.) The energy density is proportional to the space-time mean-square pressure, as we found in Eq. III.7.8b); we have assumed it to be constant. The total pressure is the sum of direct and reverberant components; simple calculations lead to:

$$\begin{aligned} \langle p^2\rangle_t &= \langle p_d^2\rangle_t + \langle p_R^2\rangle_t \\ &= \rho_o c\Pi_s\left(\frac{1}{4\pi r^2} + \frac{4}{R}\right) \quad . \end{aligned} \tag{III.9.5}$$

The pressures in the direct field and the reverberant field are equal at a distance

$$r = (R/16\pi)^{1/2} \tag{III.9.6}$$

from the source. This distance is small compared with the overall dimensions of the room, if the absorption coefficient is small and the room is therefore reverberant. Some rough calculations will illustrate this. If α is small, $R \approx S\alpha$. In typical rectangular rooms, the surface area is about

$$S \approx 6\, V^{2/3} \equiv 6L^2$$

where $L \equiv V^{1/3}$ is a characteristic room dimension. (Compare L with the mean free path, Eq. III.7.11.) In a typical, moderately reverberant room that has no special absorptive treatment, $\alpha \approx 1/6$. Then the distance for equality of direct and reverberant fields is $r \approx L/7$; the direct field predominates at points closer to the source.

Reverberation and Decay

When the sound source has been turned off, there is no longer any direct field; the energy of the reverberant field decays slowly because of absorption at the walls. This situation is described by the power-balance equation, III.9.3, with $\Pi_S = 0$. The solution is an exponential decay of energy density,

$$E_R = \Big[E_R\Big]_{t=o} e^{-\alpha \nu t} \quad , \qquad \text{(III.9.7)}$$

with a decay rate $\alpha\nu$ nepers per unit time.

This equation is formally identical with that governing the natural decay of the energy of a damped simple resonator, Eq. II.4.6. (The different use there of the symbol α is unlikely to lead to confusion.) As in the earlier case (Eq. II.4.8c), a common measure of the decay rate is the reverberation time T_R, in which interval the energy decays to $1/10^6$ of its original value (i.e. a change of 60 dB):

$$T_R = 13.8/\alpha\nu \quad . \qquad \text{(III.9.8)}$$

Modal Description for Room with Absorption

In this subsection, we sketch the manner in which the natural decay of sound in a practical room, with absorption, can be described in terms of the response of many natural modes for the room.*

*Detailed discussions are given by P. M. Morse and R. H. Bolt, "Sound in Rooms," Reviews of Mod. Phys., 16, 2 (1944) and by P. M. Morse, Vibration and Sound, (McGraw-Hill Book Company, New York, 1948), Chapter 8.

The procedure is as follows. The sound pressure in the room is expressed as a superposition of components which are natural modes of the room without absorption. The effects of small absorption are introduced, ad hoc, by the inclusion of small dissipative terms. An expression for the space-average of sound energy density is derived from the modal components of pressure. The result is compared with that derived from the approximate theory.

The total sound pressure $p(\underline{r},t)$ can be expressed as a series in the natural modes

$$p(\underline{r},t) = \Sigma_N p_N(\underline{r},t) = \Sigma_N P_N(t)\ \psi_N(\underline{r}) \quad , \qquad \text{(III.9.9)}$$

where the natural mode functions satisfy Eq. III.7.1b:

$$\nabla^2\ \psi_N + (\omega_M^2/c^2)\psi_N = 0 \quad .$$

We choose scale factors so that the spatial average of ψ_N^2 is unity:

$$\langle\psi_N^2\rangle_{\underline{r}} = 1 \quad ;$$

different modes are spatially orthogonal in the sense that

$$\langle\psi_M\psi_N\rangle_{\underline{r}} = 0\ , \qquad M \neq N \quad .$$

(Compare Eq. III.7.7c and following discussion.)

When Eq. III.9.9 is substituted into the wave equation, III.2.6, and the differential equation for ψ_N is used, one finds

$$\Sigma_N\psi_N(\underline{r})\left[\omega_N^2\ P_N(t) + d^2P_N/dt^2\right] = 0 \quad .$$

If this series is multiplied by any particular $\psi_M(\underline{r})$ and averaged over position, only a single term is left:

$$\frac{d^2P_M}{dt^2} + \omega_M^2\ P_M = 0 \quad . \qquad \text{(III.9.10)}$$

This is the differential equation for natural vibration of an undamped simple resonator with natural frequency ω_M (see Chapter II, Section 4). In the real room, energy will be absorbed at the walls and the natural motion will be damped. We assume, for small damping, that the effect of absorption is adequately reflected by the ad hoc introduction of a dissipative term in each differential equation:

$$\frac{d^2 P_M}{dt^2} + \omega_M \eta_M \frac{dP_M}{dt} + \omega_M^2 P_M = 0 \quad , \qquad \text{(III.9.11)}$$

and by the neglect of dissipative coupling between modes.

Determination of the modal loss factor η_M from the physical properties of absorptive materials in the room can be a very complicated affair. In essence, η_M can be determined from the fraction of total modal energy that is dissipative in each cycle of vibration (cf. Eq. II.4.7).

However, it is the form of Eq. III.9.11, rather than the values of the parameters, that interests us at this point. This is the differential equation for natural vibration of a damped simple resonator with natural frequency ω_M and loss factor η_M. The response of each mode is equivalent to the response of a separate, damped simple resonator. (This equivalence is valid also for steady-state, forced response of the room; compare Chapter II, section 6.)

Let us look at the energy of the sound vibration. The total energy of sound is the sum of the energies of the different modes, each of which behaves as a simple, damped resonator. We use short-time averaging to eliminate fluctuations with the natural frequency ω_M while revealing the gradual decay (compare Eq. II.4.6). We write the modal energy as the product of room volume V and the spatial average of energy density E_M. Thus, the decay of modal energy is described by the equation

$$V\langle E_M\rangle_{\underline{r},t} = \left[V\langle E_M\rangle_{\underline{r},t}\right]_{t=o} e^{-\omega_M \eta_M t} \quad . \qquad \text{(III.9.12)}$$

Now, it is found as both an experimental and a theoretical fact that all the modes resonant at about the same frequency have about the same value of loss factor and decay rate. The theoretical explanation is based on a result found above in section 7. The typical mode is an oblique one, with a k-vector which is not parallel to any walls; therefore each mode tends to "sample" and average the absorption on all the walls. If we assume that this is true, and set $\omega_M \eta_M = \omega\eta$ for all modes, then the total energy density must decay according to the same law:

$$\langle E \rangle_{\underline{r},t} = \left[\langle E \rangle_{\underline{r},t} \right]_{t=o} e^{-\omega\eta t} \quad . \qquad \text{(III.9.13)}$$

This result is in the same form as Eq. III.9.7, derived from the approximate theory. The equivalence of the average boundary absorption coefficient α and average modal loss factor η is found by equating the decay rates:

$$\alpha\nu = \omega\eta \quad . \qquad \text{(III.9.14a)}$$

This can be rewritten in terms of the mean free path $d = c/\nu$ (Eq. III.7.11):

$$\alpha = kd\eta \text{ , where } k = \omega/c \quad . \qquad \text{(III.9.14b)}$$

The result is a useful estimation formula, since d equals a "typical" dimension of the room.

IV. WAVES IN STRUCTURES

IV.1 Wave Types in Structures

As we have seen, the forms of vibration in rooms are very complicated, due to the very many degrees of freedom (normal modes) which may be excited and the complicated pressure pattern of each. In addition, the boundary conditions are complicated by the geometrical irregularity of the bounding surfaces and the lack of understanding of the exact pressure-velocity relations which must be satisfied at the boundaries. Nevertheless, the acoustic field is relatively simple in comparison with the multi-modal vibrational patterns possible on plates and beams.

In detail, any structure is a three-dimensional elastic (or viscoelastic) medium capable of supporting the fundamental shear and compressional wave types. Analyses of plates and beams indicate however, that when the wavelengths are long compared to the cross-dimensions of the beam or plate, then the motions may be analyzed into longitudinal (compressional) and transverse (bending) wave types. In this section, we shall only consider bending or flexural motions. This is because the transverse motions of structures will normally couple more strongly to the sound field and because there are many more flexural modes in structures at the lower frequencies than there are longitudinal ones.

IV.2 Equations of Motion for Bending Waves

In this section we shall derive equations of bending motion for waves in a thin rod (beam). The equations will not be derived for a plate, but will be stated as an analogy to the one-dimensional case. Consider an element of the beam as shown in Fig. IV.1. The intersections of the neutral axis of the beam with the two imaginary faces of the element are at A and B. Upon motion of the beam these points move to C and D respectively. The angle between the horizontal and the neutral axis at C is θ_x and at D it is $\theta_{x+\Delta x}$. The beam element will be bent as shown in the figure if θ_x is greater than $\theta_{x+\Delta x}$. The fiber at a distance z from the neutral axis will undergo an elongation

$$\Delta \ell = - z \frac{\partial \theta}{\partial x} \Delta x \quad , \qquad \text{(IV.2.1)}$$

where $\Delta \ell$ is the change in length and Δx is the original length of the element.

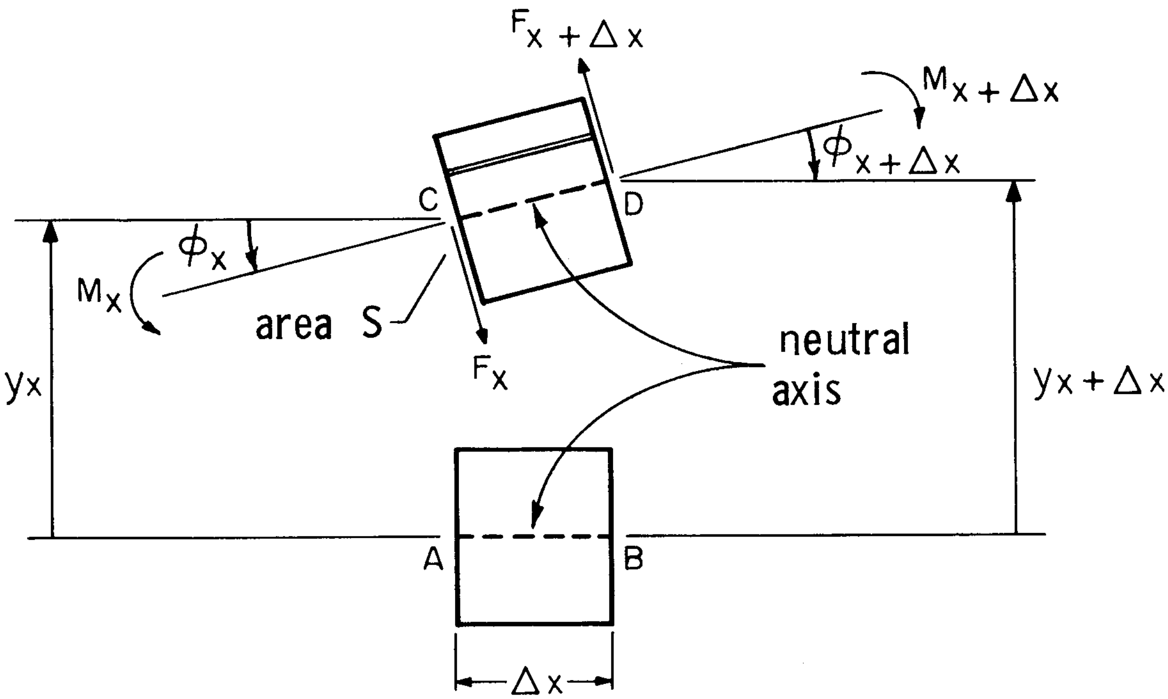

Figure IV.1. - Dynamics of beam element.

The fiber is assumed to have an area dS so that the element of force required for its stretching is

$$df = Y_o \frac{\Delta \ell}{\Delta x} dS = - Y_o z \frac{\partial \theta}{\partial x} dS \quad . \tag{IV.2.2}$$

If the sign convention for moments is as indicated on the figure, then the moment applied to the left-hand face is

$$M_x = \int s df = - Y_o \frac{\partial}{\partial x} \int_S z^2 dS \tag{IV.2.3}$$

where the integration has taken over the surface S. For small deflections $\theta \simeq \frac{\partial y}{\partial x}$ and defining the radius of gyration, κ, by $\kappa^2 = \frac{1}{S}\int z^2 dS$, we can write the moment as

$$M_x = - Y_o S \kappa^2 \frac{\partial^2 y}{\partial x^2} \quad . \tag{IV.2.4}$$

If we neglect the effects of rotatory inertia a shear force will be established at the faces of the element which will counter-balance the applied moment to the element. The net clock-wise moment is $\frac{\partial M}{\partial x} \Delta x$. The shear force is therefore,

$$F = \frac{\partial M}{\partial x} = - Y_o S \kappa^2 \frac{\partial^3 y}{\partial x^3} \tag{IV.2.5}$$

and the net vertical force on the element is

$$\frac{\partial F}{\partial x} \Delta x \cos \theta \simeq Y_o S \kappa^2 \frac{\partial^4 y}{\partial x^4} \Delta x \tag{IV.2.6}$$

where we have assumed that $\cos \theta$ is essentially unity.

If there is an applied vertical force of strength f_ℓ per unit length then the total vertical force is the sum of f_ℓ and the elastic forces computed above. This force must equal the mass of the element times the vertical acceleration $\frac{\partial^2 y}{\partial t^2}$. Combining these, the equation of motion for the beam element is

$$\rho_m S \frac{\partial^2 y}{\partial t^2} + Y_o S \kappa^2 \frac{\partial^4 y}{\partial x^4} = f_\ell(x,t) \qquad \text{(IV.2.7)}$$

where ρ_m is the density of the beam material. The analogous equation for two dimensions is derived in Love [Mathematical Theory of Elasticity, Dover Publications, Inc., New York 1944, p. 488] and is

$$\rho_m h \frac{\partial^2 y}{\partial t^2} + Y_p h \kappa^2 \nabla^2\nabla^2 y = f(\underline{x},t) \qquad \text{(IV.2.8)}$$

where $Y_p = Y_o/(1-\sigma^2)$ and σ is Poisson's ratio. The plate thickness is h.

Like the acoustic case, the damping in structural elements arises from a combination of sources. Empirical evidence suggests that the greatest amount of damping in built-up structures occurs at the structural joints and connections between the elements rather than in the inherent damping associated with material hysteresis and/or plastic deformation. In some cases the acoustic radiation damping of a structure may become important in its damping also.

The detailed inclusion of the damping phenomena in the equations of motion is not possible or practical on two counts. The first is the largely unknown nature of the damping at structural joints. The second is the very complicated fashion in which the damping would enter the equations. We shall instead borrow the procedure we followed in the acoustic case and assume a mathematically convenient form of damping which will subsequently be related to experimental measurements of modal bandwidths and reverberation time.

Damping is usually introduced in the equations of motion in one of two fashions. In the first it is added as a simple viscous resistance term

$$\frac{\partial^2 y}{\partial t^2} + \eta\omega \frac{\partial y}{\partial t} + \kappa^2 c_\ell^2 \nabla^4 y = \frac{f}{\rho_m h} \qquad \text{(IV.2.9)}$$

where $c_\ell^2 = Y_p/\rho_m$ is the square of the velocity of longitudinal waves in the plate and η is the structural loss factor. Alternatively, if one assumes a time dependence $e^{i\omega t}$ then the damping may be incorporated as a complex Young's modulus $Y_p(1-i\eta)$ to give

$$\frac{\partial^2 y}{\partial t^2} + \kappa^2 c_\ell^2 (1+i\eta)\nabla^4 y = \frac{f}{\rho_m h} \quad . \tag{IV.2.10}$$

As we shall see these are essentially equivalent in most calculations.

IV.3 Solutions of the Bending Equation

The general two-dimensional solution of Eq. IV.2.8 has not been achieved. Nevertheless, a substantial amount of information concerning bending waves can be gained by solving for the one-dimensional solutions assuming harmonic time dependence $e^{i\omega t}$. Without damping and external forces then Eq. IV.2.8 assumes the form

$$\kappa^2 c_\ell^2 \frac{d^4}{dx^4} y - \omega^2 y = 0 \quad . \tag{IV.3.1}$$

Since this is a simple linear differential equation with constant coefficients, we assume a solution of the form $y \sim e^{\lambda x}$. This gives a solution for λ:

$$\lambda^4 = \frac{\omega^2}{\kappa^2 c_\ell^2} \; ; \qquad \lambda = \pm\, k_b \quad , \quad \pm\, i k_b \tag{IV.3.2}$$

where $k_b = (\omega/\kappa c_\ell)^{1/2}$.

The general solution to Eq. IV.3.1 is therefore

$$y = \left[a e^{-ik_b x} + b e^{ik_b x} + c e^{-k_b x} + d e^{k_b x} \right] e^{i\omega t} \tag{IV.3.3}$$

where a, b, c, and d are coefficients to be evaluated depending on the boundary conditions. As we discussed in the previous chapter, the first two terms in the solution represent traveling waves in the +x and -x directions respectively. The last two terms represent "near-field" terms which can exist near the boundaries of the plate (or near the source). The solutions Eq. IV.3.3 are also applicable to a beam if we interpret the longitudinal wave speed c_ℓ to be $(y_0 \rho_m)^{1/2}$, the values appropriate to the beam. The longitudinal speed in plates and beams of the same material and thickness will only differ by 5%. We shall not consistently make the distinction in the discussion to follow.

Let us first consider the solution for an infinite beam. In this case c and d must vanish if the solutions are to remain finite. The wave in the +x direction therefore propagates with a phase speed $c_b = \omega/k_b = (\omega\kappa c_\ell)^{1/2}$. Unlike the acoustic wave the bending wave has a phase speed which increases as the square root of the frequency. It is this frequency dependence of speed, a so-called dispersion effect, which is central to many of the interesting properties of beams and plates both in regard to their structural behavior and to their coupling to a sound field. This simple dispersion law is valid until the bending wavelength becomes approximately six times the plate thickness h. For shorter wavelengths or higher frequencies a more complete analysis is required such as that developed by Mindlin [R. D. Mindlin, J. Appl. Mech., Vol. 18, p. 36; March 1951]. In this chapter, however, we shall assume that the simple bending equations are adequate to describe the transverse flexural movement of structures.

IV.4. Energy Transport in Flexural Wave Motion

We consider a simple propagating wave in the +x direction $y = Ae^{-ik_b x}$. The velocity associated with this displacement wave is $v + i\omega Ae^{-ik_b x}$. The time average kinetic energy at any position is $1/4\rho_m S|v|^2$. In a propagating wave the average kinetic energy is everywhere the same. The total average energy that is twice this since the kinetic and potential energies are equal on the average. The total energy is

$$E = \frac{1}{2}\rho_m S|v|^2 = \frac{1}{2}\rho_m S\omega^2|A|^2 \quad . \tag{IV.4.1}$$

The transverse shear force associated with the wave is

$$F = -Y_o S\kappa^2 \frac{d^3y}{dx^3} = -ik_b^3 Y_o S\kappa^2 Ae^{-ik_b x} \tag{IV.4.2}$$

and the time average force velocity product at any point is, therefore,

$$\langle \mathrm{Re}(F)\,\mathrm{Re}(v)\rangle_t = \frac{1}{2}\mathrm{Re}(Fv^*) = \frac{1}{2}k_b^3 Y_o S\kappa^2\omega|A|^2 \quad . \tag{IV.4.3}$$

This is the rate at which work is done by one part of the beam or the other due to the passage of the wave through the shear forces. There is another part of work done by the moment forces acting through the angular velocity $\frac{\partial}{\partial t}\frac{\partial y}{\partial x}$. The rate at which work is done by the moment is therefore

$$\langle Re(M)Re(\frac{\partial\theta}{\partial t})\rangle_t = \langle Re(i\omega\frac{\partial y}{\partial x}) \cdot Re(Y_o S\kappa^2 \frac{d^2y}{dx^2})\rangle_t = \langle Re(F)Re(v)\rangle_t \tag{IV.4.4}$$

The rates of work done by the moment and by the shear force are therefore equal and the total transport of energy across any position is just

$$\Pi = Y_o S\kappa^2\omega|A|^2k_b^3 \quad . \tag{IV.4.5}$$

The energy velocity is defined as the ratio of the power to the energy density

$$c_E \equiv \Pi/E = 2(\omega\kappa c_\ell)^{\frac{1}{2}} = 2c_b \quad . \tag{IV.4.6}$$

We note here another difference from the acoustic case. The wave dispersion has caused the energy and the wave fronts to travel at different speeds. It is interesting to note that the group velocity* for the dispersive wave is

$$c_g = 1/\frac{dk}{d\omega} = 2c_b \tag{IV.4.7}$$

and is also equal to twice the phase speed. It makes sense that the group velocity and the energy velocity should be the same since the group velocity is the speed with which the envelope of a wave packet will propagate along the beam. The packet of waves contains a clump of energy which is being propagated along at the speed c_g.

*For example J. J. Stoker, Water Waves (Interscience Publishers, Inc., New York, 1957) p. 51.

IV.5 Modes of Vibration in One and Two-Dimensional Structures

In much of the study of the interaction of sound with a structure, it is convenient to consider the structure as a collection of natural modes of vibration. As we shall see, the number of cases for which these modes are known exactly are very few. It is possible to generate some useful conclusions about their behavior by the consideration of a few simple cases which can be solved. The simplest of these is a beam which is supported at both of its ends.

We wish to find the equations for mode shape and frequency of natural vibration of the beam shown in Fig. IV.2. In the absence of damping and external forces, the equation of motion for the beam is given in Eq. IV.3.1 and the general form of its motion is given in Eq. IV.3.3. By proper combination of the exponents we can rewrite Eq. IV.3.3 in the general form.

$$y = A \sin k_b x + B \cos k_b x + C \sinh k_b x + D \cosh k_b x \tag{IV.5.1}$$

where A, B, C, and D are coefficients related to a, b, c, and d. It will turn out that Eq. IV.5.1 is a more convenient form of the response to use than Eq. IV.3.3 when we are dealing with standing waves.

We assume an ideal simple support at the ends $x = 0$, and $x = \ell$. Such a support applies only a transverse force at the ends so that there is no resulting transverse displacement or moment applied. The boundary conditions at $x = 0$ and $x = \ell$ are accordingly

$$y = 0, \; \frac{d^2y}{dx^2} = 0 \;@\; x = 0, \ell \quad . \tag{IV.5.2}$$

The requirement that y vanish at $x = 0$ requires that $B + D = 0$. The requirement that the second derivative should vanish with the origin means that $B - D = 0$. These two conditions taken together require that B and D must vanish individually. Eliminating the circular and hyperbolic cosine functions and requiring that y vanish at $x = \ell$ gives

$$A \sin k_b \ell + C \sinh k_b \ell = 0 \tag{IV.5.3}$$

Figure IV.2.- Configuration of supported beam.

while requiring that the second derivative vanish there results in

$$- A \sin k_b\ell + C \sinh k_b\ell = 0 \quad . \tag{IV.5.4}$$

The two conditions Eq. IV.5.3 and Eq. IV.5.4 require that both A sin $k_b\ell$ and C sinh $k_b\ell$ should vanish. Since the hyperbolic sine only vanishes at the origin, this means that C = 0. The conditions are also satisfied by A = 0, but that is a trivial situation not involving motion of the beam. One can only have non-trivial solutions, therefore, if sin $k_b\ell = 0$ or if $k_b\ell = m\pi$. The eigenfunctions for the simply supported beam are therefore,

$$\psi_m = 2^{1/2} \sin m\pi x/\ell \quad , \tag{IV.5.5}$$

normalized as in the acoustic case such that $\langle\psi_m^2\rangle = 1$.

The natural frequencies of vibration are found from the general relationship Eq. IV.3.2 so that

$$\omega_m = k_b^2 \kappa c_\ell = \frac{m^2\pi^2}{\ell^2} \kappa c_\ell \quad . \tag{IV.5.6}$$

Note that as a result of the dispersive nature of the beam, the natural frequencies proceed as the square of the integers rather than the integers themselves as is the case for sound waves. This has an important effect on the distribution of energy in frequency for one-dimensional structures.

<u>Modal Density</u>

The values of k_b which correspond to the natural modes may be displayed on a linear plot as shown in Fig. IV.3. The number of modes below any value k is given by

$$N(k) = k/\tfrac{\pi}{\ell} = k\ell/\pi \quad . \tag{IV.5.7}$$

In the average number of modes per interval N(k) is therefore

$$n(k) = \frac{dN(k)}{dk} = \frac{\ell}{\pi} \quad . \tag{IV.5.8}$$

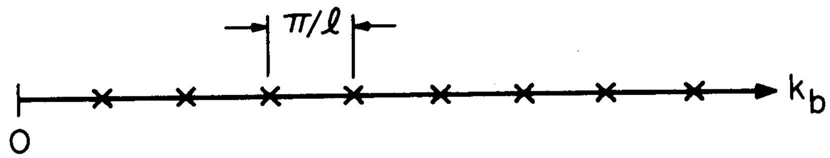

Figure IV.3.- Modal pattern in "k-space" for supported beam.

The modal density in frequency is found by the simple transformation

$$n(\omega) = n(k)\,\frac{dk}{d\omega} = \frac{\ell}{\pi c_g} \quad . \tag{IV.5.9}$$

Up to this point our result is completely general. If one places the value of group velocity found in Eq. IV.4.7 into IV.5.9 one has the result

$$n(\omega) = \frac{\ell}{2\pi\sqrt{\omega\kappa c_\ell}} \tag{IV.5.10}$$

which decreases as $\omega^{1/2}$. At this point it is clear that the geometry and dynamics of a mechanical system will have a very great effect on its modal density. For the room we found that the density of modes increased as ω^2 resulting in a very great density of modes in the audio frequency range for rooms of typical size. Eventually, at higher frequencies where finite thickness effects become important, the beam will also have an increasing modal density. However, in the lower frequency region where bending wave equation describes its dynamics, its modal density is relatively low and decreases with increasing frequency.

Beam Clamped at Both Ends

A beam which has both of its ends held in place by vise-like clamps is shown in Fig. IV.4. In this case both transverse forces and moments are applied to the ends of the beam so that the displacement and the slope both vanish there. The boundary conditions for the clamped-clamped beam are

$$y = 0,\ \frac{dy}{dx} = 0 \ @\ x = 0, \ell \quad . \tag{IV.5.11}$$

As before, three of the constants and the allowed values of wave number are determined by these four boundary conditions. The fourth constant is determined by the normalization. We shall not go through the derivation for this case since the results are more complicated and involved considerably more algebra than before, because the hyperbolic functions are not eliminated. The details of the derivation are carried out by Morse* and here we merely state the result for the allowed wave numbers. The equation for the wave numbers is

*P. M. Morse, Vibration and Sound, 2nd Ed., (McGraw-Hill Book Company, Inc., New York, 1948) pp. 156f.

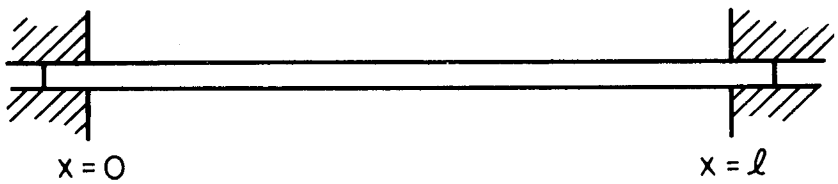

Figure IV.4.- Configuration of clamped-clamped beam.

$$\cos k_b \ell = 1/\cosh k_b \ell \quad . \qquad \text{(IV.5.12)}$$

The two sides of this equation are plotted in Fig. IV.5. The intersections of the two curves represent solutions for the allowed values of $k\ell$. As we see, they are $k\ell = 0$ and $k\ell \approx (m+1/2)\pi$. These values of $k\ell$ are plotted on Fig. IV.6. Note that at higher wave numbers the spacing of modes is the same as in Fig. IV.3. Thus, although the values of resonance wave numbers are different, the modal density for the clamped and the supported beams become equal above the first few modes of vibration. The resonance at $k = 0$ is one of uniform motion, but the resulting amplitude for a clamped-clamped beam is zero.

The free-free beam, for which there is no moment or shear force applied to either end, has a frequency equation identical to that of Eq. IV.5.12. The modal shape functions are different, but the allowed frequencies of vibration are the same as for the clamped beam. In this case the value of the wave number $k_b = 0$ corresponds to rigid body translation of the beam.

Vibration of a Simply Supported Rectangular Plate

Consider the rectangular plate of dimensions ℓ_1, ℓ_2 shown in Fig. IV.7. We assume that the plate has simply supported edges like those shown in Fig. IV.2. The general boundary conditions for a plate are more complex than for the one-dimensional beam which we have derived. The general boundary conditions, which are given in Love,* reduce to the simple one-dimensional condition when applied to a plate with a rectangular edge. Therefore, for the simply supported case the boundary conditions are

$$\left.\begin{aligned} y = 0, \quad \frac{\partial^2 y}{\partial x_1^2} = 0 \; ; \quad @ \quad x_1 = 0, \ell \\ y = 0, \quad \frac{\partial^2 y}{\partial x_2^2} = 0 \; ; \quad @ \quad x_2 = 0, \ell \end{aligned}\right\} \quad . \qquad \text{(IV.5.13)}$$

*A. E. H. Love, Mathematical Theory of Elasticity (Dover Publications, Inc., New York, 1944) p. 488.

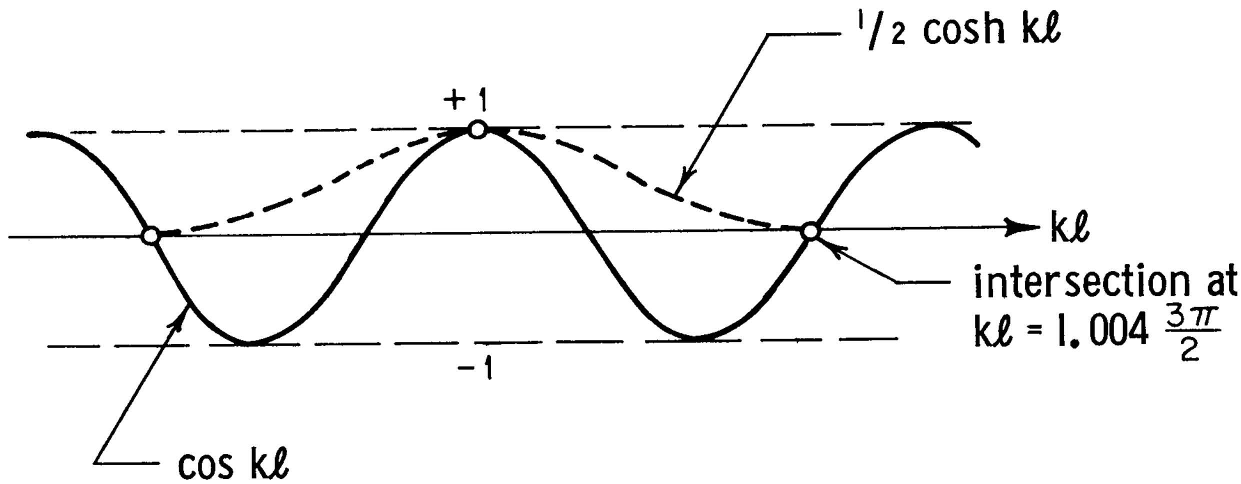

Figure IV.5.- Evaluation of resonance wavenumbers for clamped beam.

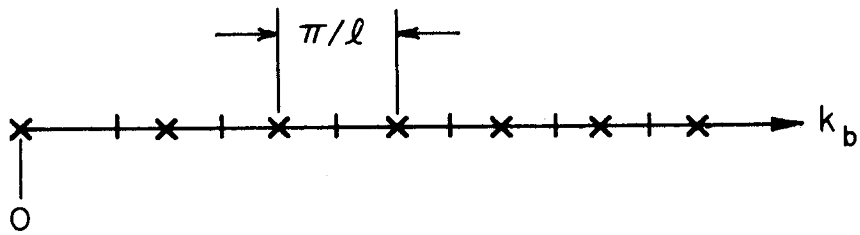

Figure IV.6.- Modal pattern in "k-space" for clamped beam.

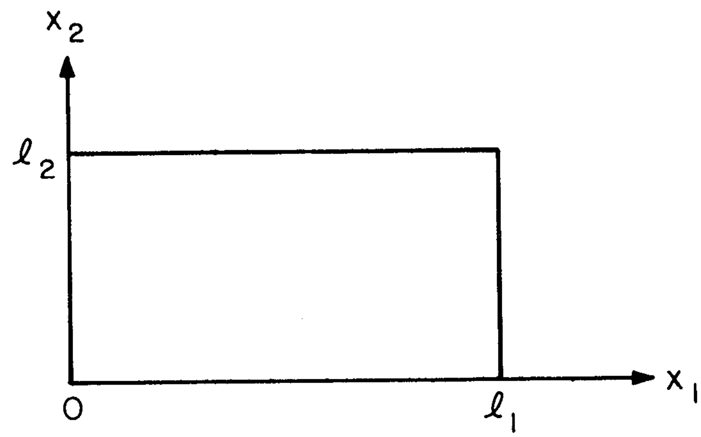

Figure IV.7.- Geometry of simply supported rectangular plate.

The solution of Eq. IV.2.9 for harmonic time dependence and no damping or excitation is known to be

$$y \sim \sin k_1x_1 \sin k_2x_2 \quad , \tag{IV.5.14}$$

the product of simply supported beam modes. The reader will note that we have not derived a solution of Eq. IV.2.9 in the sense that we derived the solution for the one-dimensional beam. We have merely stated that the solution Eq. IV.5.14 satisfies the boundary condition and, if substituted into the equation of motion, will yield a consistency relation for the wave number

$$k_1^2 + k_2^2 = k_p^2 = \frac{m^2\pi^2}{\ell_1^2} + \frac{m^2\pi^2}{\ell_2^2} \quad . \tag{IV.5.15}$$

By contrast, one can meet all the boundary conditions for a plate with fully clamped edges by forming a product of the natural mode functions for the one-dimensional beam. In this case, however, the product function does not satisfy the equation of motion Eq. IV.2.9. In fact, it is generally true that there exists no separable solution, nor any solutions in terms of simple functions, for the two-dimensional bending equation. A great deal of effort has gone into the generation of efficient and appropriate approximation schemes for determining the eigenfunctions of two-dimensional plate structures.

We can develop a fair amount of information about structures from the simply supported rectangular plate just as we could derive information about the general properties of three-dimensional acoustic spaces in Chapter III by considering the simple rectangular parallel piped room. In Fig. IV.8 we have drawn the modal lattice derived from Eq. IV.5.15. As we discussed in Chapter III, the distance from the origin to a lattice intersection point is the wave number k_p and depends on the frequency through the usual relationship. The resonant frequency corresponding to the modes is

$$\omega_m = \kappa c_\ell \, k_p^2 = \pi^2 \, \kappa c_\ell \left(\frac{m^2}{\ell_1^2} + \frac{n^2}{\ell_2^2}\right) \quad . \tag{IV.5.16}$$

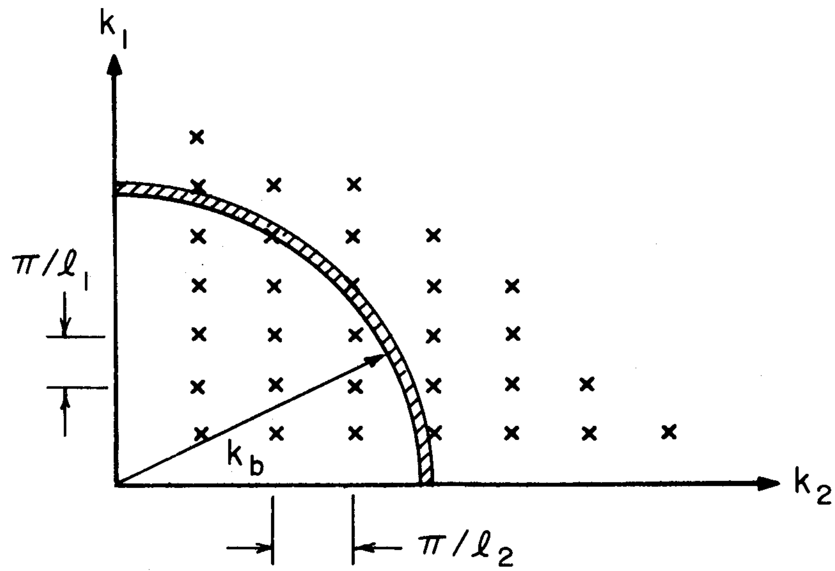

Figure IV.8.- Modal lattice and constant frequency contours for supported plate.

Modal Density

The modal density for a plate is found by considering the area included by the quarter circle of radius k_p shown in Fig. IV.8. This area is $\pi k_p^2/4$. The area which may be assigned to each mode is $\pi^2/\ell_1\ell_2$. The number of modes included up to a wave number k_p is, therefore,

$$N(k) = \frac{k^2\ell_1\ell_2}{4\pi} \tag{IV.5.17}$$

and the modal density in k is

$$n(k) = \frac{k\ell_1\ell_2}{2\pi} \quad . \tag{IV.5.18}$$

Using $n(\omega) = n(k)/c_g$, one has for the modal density in frequency

$$n(\omega) = \frac{k\ell_1\ell_2}{4\pi c_b} = \frac{A_p}{4\pi\kappa c_\ell} \tag{IV.5.19}$$

where we have used $A_p = \ell_1\ell_2$, and $k = \omega/c_b$. The remarkable thing about Eq. IV.5.19 is that the modal density is frequency independent. Although we have derived this for a simply supported plate, analogy with the previous situations we have studied suggests that it is a general result that the modes for real systems will occur somewhat irregularly but will have a uniform modal density on the average.

IV.6 Energy Reverberation in Two-Dimensional Structures

In this section we parallel some of the discussions of the reverberation of energy in rooms at frequencies well above the first mode so that many wave fronts and modes participate in the motion. To do this we study the concepts of mean-free path, modal energy, and energy dissipation as they apply to two-dimensional structures.

Mean-Free Path for Two-Dimensional Structures

The average rate at which the energy of panel motion will encounter the panel boundaries can be computed by a process similar to that used for the reverberation of the sound in rooms in section III.7. Consider for example an irregularly shaped panel like that shown in Fig. IV.9. A certain fraction of the energy will travel in a direction indicated by the arrows in the figure. Let this direction be defined by the angle it makes with the normal to a small element of the perimeter $d\ell$. The rate of energy incident on the edge, Π, for waves in this direction depends on the intensity I and the projected perimeter $\Lambda(\theta)$ indicated in Fig. IV.9

$$d\Pi(\theta) = dI\Lambda(\theta) \quad . \tag{IV.6.1}$$

For one-dimensional waves the intensity and energy are related by the energy velocity as shown in Eq. IV.4.6. The energy density of the wave is, therefore

$$dE = dI/c_E \quad . \tag{IV.6.2}$$

If the incident intensity is uniformly distributed in direction, one has $dI = Id\theta/2\pi$. The average collision rate ν was defined previously as the ratio of power incident on the boundary to the energy contained in the panel;

$$\begin{aligned} \nu &= \int_\theta d\Pi(\theta) \Big/ A_p \int_\theta dE(\theta) \\ &= I\langle\Lambda(\theta)\rangle_\theta \Big/ \frac{A_p I}{c_E} \quad . \end{aligned} \tag{IV.6.3}$$

The average value of the projected perimeter Λ is found by integrating the projected lengths of the elements $d\ell|\cos\theta|$ through which the energy "leaves" the structure. Averaging this over the directions corresponding to the energy exit, we have

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} d\ell|\cos\,\theta|d\theta = \frac{1}{\pi}\,d\ell \quad . \tag{IV.6.4}$$

Integrating this average projected element length over the entire perimeter gives

$$\langle\Lambda(\theta)\rangle_\theta = \int_L \frac{1}{\pi}\,d\ell = \frac{1}{\pi}\,L \tag{IV.6.5}$$

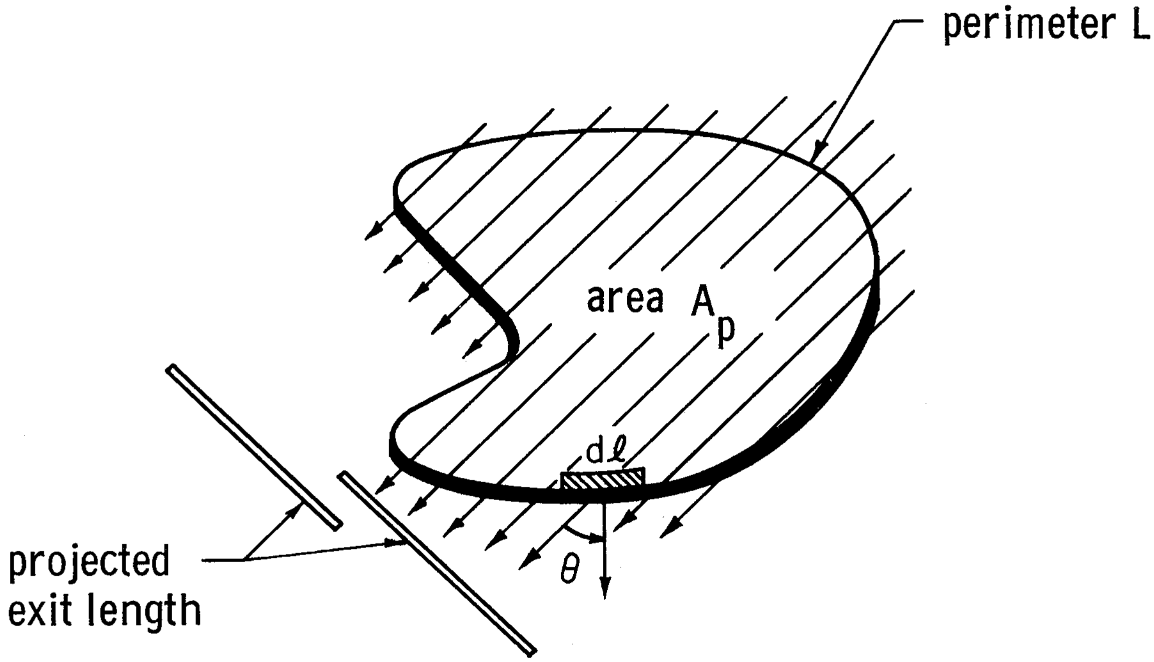

Figure IV.9.- Mean free path computation for two-dimensional structures.

$$\nu = \frac{c_E L}{\pi A_p} \quad . \tag{IV.6.6}$$

If, as in the acoustic case we define the mean-free path d as the ratio of the energy velocity to the collision rate ν, we have

$$d = \pi A_p / L \quad . \tag{IV.6.7}$$

This is a general result for two-dimensional systems.

Energy Absorption in Reverberant Panels

The absorption of energy in mechanical structures is frequently described by the loss factor η. In built-up structures the losses that a panel experiences are usually associated with its boundaries, either because the energy is propagated through the boundary to other parts of the structure or because dissipation processes take place there due to interfaces between the panel and its supporting frames. It has, therefore, become convenient to think of the edges as possessing a certain absorption coefficient γ which is similar to the acoustic absorption coefficient for walls described in Chapter III. The relation between absorption coefficient and loss factor is a useful one for relating experimental decay curves to absorption mechanisms at the boundaries.

The discussion proceeds in a manner entirely analogous to that in Chapter III. If the input power to a point source of mechanical excitation is Π_s then the power entering the reverberant field is $\Pi_s(1-\gamma)$. This power enters the reverberant field after one collision with the boundaries as shown in Fig. IV.10. From our previous discussion the average rate of energy encounter with the boundaries is $E_R A_p \nu$ where E_R is the average energy density of the reverberant field. If a fraction γ is lost upon each encounter then the energy balance equation is

$$\Pi_s(1-\gamma) = E_R A_p \nu \gamma \tag{IV.6.8}$$

resulting in an equilibrium energy in the reverberant field

$$E_R = \frac{\Pi_s}{\nu A_p} \frac{\gamma}{1-\gamma} \quad . \tag{IV.6.9}$$

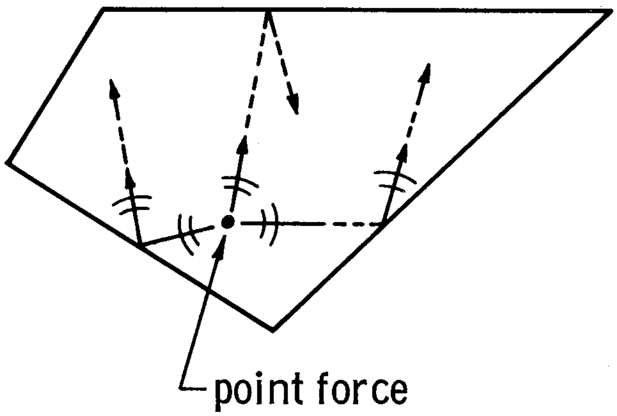

Figure IV.10.- Direct and reverberant field inputs from a point source.

The loss factor is defined by

$$\Pi_{diss} = \omega\eta E_R A_p \tag{IV.6.10}$$

which requires $\omega\eta = {}^{\nu}\gamma$. Recalling that the energy velocity is twice the phase speed,

$$\gamma = \frac{1}{2} k_b \eta d \quad . \tag{IV.6.11}$$

We note that even if the absorption coefficient is unity one can have resonant modes corresponding to $\eta << 1$ if the mode order is high, i.e., $k_b d >> 1$.

Panel Energy and Kinetic Velocity

The kinetic energy of any small element of the panel is given by

$$dT = \frac{1}{2} \rho_m h \, dS \langle v^2 \rangle_t \tag{IV.6.12}$$

where v is the real transverse velocity of the element. The total kinetic energy is obtained by integrating this over the panel area

$$\langle T \rangle_{x,t} = \frac{1}{2} \rho_m h \, A_p \langle v^2 \rangle_{t,x} \quad . \tag{IV.6.13}$$

For each mode of vibration, the average kinetic energy is equal to the potential energy and, therefore, the total energy is just twice the average kinetic energy $\langle E \rangle = 2\langle T \rangle / A_p$. The mean-square transverse velocity averaged over space and time is, therefore,

$$\langle v^2 \rangle_{t,x} = \langle E \rangle A_p / M_p \equiv v_k^2 \tag{IV.6.14}$$

where M_p is the total mass of the panel. The quantity v_k we term the kinetic velocity and it is defined by the total energy of the structure and the mass of the panel. For reverberant vibrational fields on beams and uniform panels, the kinetic velocity is identically equal to the transverse velocity averaged over the panel. In many situations, however, the kinetic velocity will not correspond to any easily measured velocity, but it may still be convenient to represent the energies stored in the panel by a velocity variable rather than the energy.

IV.7 The Input Impedance of Infinite and Finite Plates

There are many experimental situations where one would like to know the local response of the panel to an applied point force. The ratio of the local velocity response to the force is the input point admittance. The real part of this complex ratio is the input conductance and it determines the power flow into the panel from the point force. The magnitude of the admittance determines the local rms velocity. In this section we describe the manner in which this quality is calculated for infinite and finite structures. Although infinite structures are never realized, structures are effectively infinite if the reflections from the boundaries are diminished in amplitude so that they are imperceptible when they return to the point of excitation. Since perfect absorption at the boundary and absence of reverberation corresponds to γ of the order of unity, from Eq. IV.6.11 we see that this corresponds to $k_b d\eta \gtrsim 1$. The resonant character of the panel will be evident when $k_b d\eta < 1$.

Response of Infinite Plate to a Point Force

We consider an infinite plate of thickness h excited by a harmonic time force of amplitude f_o as shown in Fig. IV.11. The equation of motion with damping is given in Eq. IV.2.8 assuming harmonic time dependence;

$$[\kappa^2 c_\ell^2 \nabla^4 - \omega^2(1-i\eta)]\, y = f_o \delta(\underline{x})/\rho_m h \qquad \text{(IV.7.1)}$$

where the δ function, described in Chapter II, defines the spatial distribution of the point load.

The solution is obtained by expanding the displacement y and the force f in the two-dimensional infinite Fourier transform. The transform convention is defined by

$$Y(k) = \frac{1}{4\pi^2} \int d\underline{x}\; e^{-i\underline{k}\cdot\underline{x}} y(\underline{x}) \qquad \text{(IV.7.2)}$$

and the inverse transform is accordingly

$$y(x) = \int d\underline{k}\; e^{i\underline{k}\cdot\underline{x}} Y(\underline{k}) \quad . \qquad \text{(IV.7.3)}$$

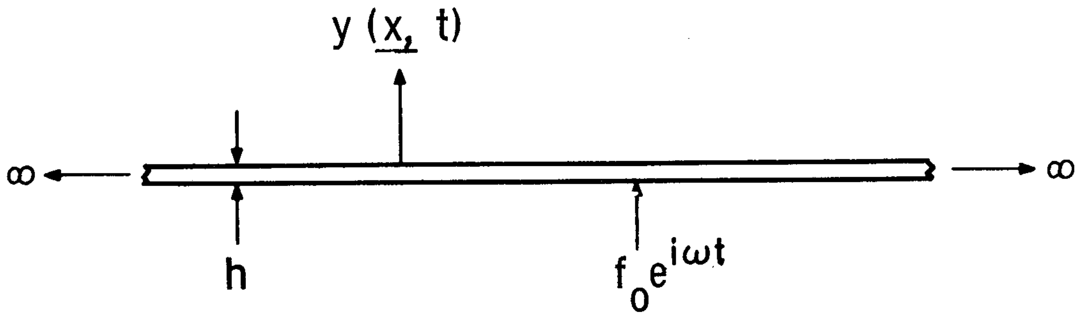

Figure IV.11.- Plate geometry for admittance calculation.

Substituting the transforms of y and f in Eq. IV.7.1, we get the transform relation

$$Y(\underline{k}) = \frac{F(\underline{k})}{\kappa^2 c_\ell^2 k^4 - \omega^2(1-i\eta)} = \frac{1}{4\pi^2} \frac{f_o/\rho_m h\kappa^2 c_\ell^2}{k^4 - k_b^4(1-i\eta)} \quad . \tag{IV.7.4}$$

Since the integrand depends only on the magnitude of k, we express Eq. IV.7.2 in terms of polar coordinates:

$$y = \int_0^\infty k dk \int_0^{2\pi} d\phi \; e^{ikr \cos \phi} \; Y(k) \quad . \tag{IV.7.5}$$

Equation IV.7.5 gives us the general response at all positions on the plate. To compute the input admittance we are only concerned about the response at the point of excitation $r = 0$. In addition we make the substitution $\zeta = k^2$, $\zeta_b = k_b^2$ and obtain

$$y(0) = \frac{1}{2} \frac{f_o}{4\pi^2 \rho_m h\kappa^2 c_\ell^2} \int_0^{2\pi} d\phi \int_0^\infty d\zeta [\zeta^2 - \zeta_b^2(1-\eta)]^{-1} \tag{IV.7.6}$$

The integral over ϕ may be carried out immediately and gives 2π. Since the integral in ζ is even, we integrate from $-\infty \rightarrow +\infty$ to give

$$y(0) = \frac{f_o}{8\pi \rho_m h\kappa^2 c_\ell^2} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta^2 - \zeta_b^2(1-i\eta)} \quad . \tag{IV.7.7}$$

The poles of the integrand in Eq. IV.7.7 and the path of integration are shown in Fig. IV.12. Closing the contour as shown, the integral has the value $-i\pi/\zeta_b$. The ratio of velocity to force at the driving point is accordingly the input admittance:

$$G_\infty = \frac{i\omega y(0)}{f_o} = \frac{1}{8\rho_m h\kappa c_\ell} \quad . \tag{IV.7.8}$$

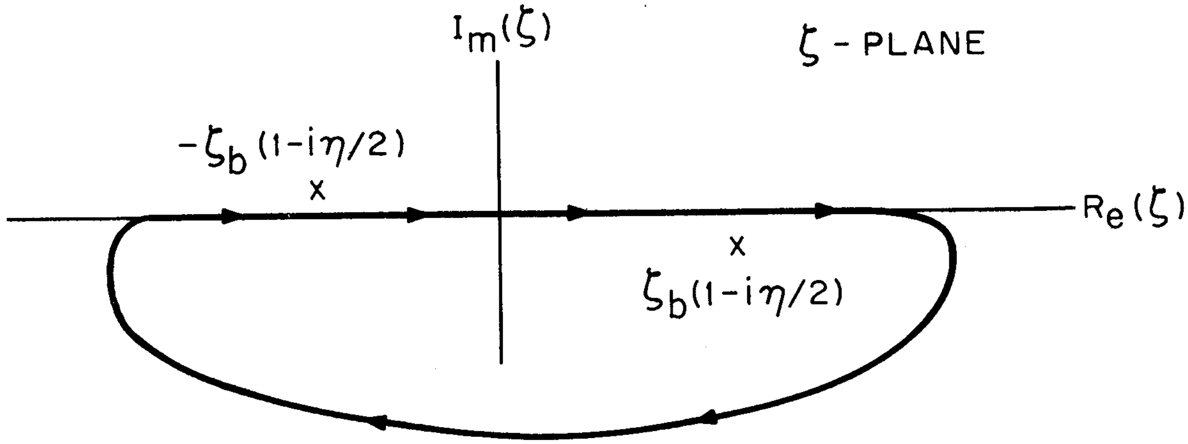

Figure IV.12.- Location of poles and integration path in ζ integration.

It is a very interesting result that the input admittance for a thin plate is purely real without frequency dependence. The frequency range for which this is true is, of course, restricted to frequencies for which the simple bending equation is valid.

Input Admittance of a Finite Plate for a Point Transverse Force

In this section we consider the local response to a point transverse force for a finite rectangular simply supported plate. The differential equation governing the motion is still given by Eq. IV.7.1 except that the source is now assumed to be located at an arbitrary position $\underline{x}_s$. We expand the displacement and the excitation in the eigenfunctions ψ_m which were determined previously.

$$\left.\begin{aligned} y &= \sum \gamma_M \psi_M \quad , \qquad \gamma_M = \langle y\psi_M \rangle_x \\ p &= \sum \rho_M \psi_M \quad , \qquad \rho_M = \langle p\psi_M \rangle_x \end{aligned}\right\} \qquad (IV.7.9)$$

If the applied pressure p is a pure tone force given by $f_o e^{i\omega t}\delta(\underline{x}-\underline{x}_s)$, using the complex convention, then

$$\rho_M = \frac{f_o}{A_p}\,\psi_M(x_s) \quad . \qquad (IV.7.10)$$

The equation governing the response amplitude y_M is

$$\left[-\omega_M^2 + \omega^2(1-i\eta)\right]\gamma_M = \rho_M/\rho_m h \quad , \qquad (IV.7.11)$$

and the general velocity response at $\underline{x}_s$ is, therefore,

$$i\omega y(x_s) = \frac{i\omega f_o}{\rho_m h A_p}\sum_M \frac{\psi_M^2(x_s)}{\omega^2-\omega_M^2+i\omega^2\eta} \quad . \qquad (IV.7.12)$$

The ratio of velocity to force is a highly fluctuating function of frequency with real and imaginary components. The real part must, of course, be positive but the imaginary part will have positive and negative fluctuations depending on the relative value of the frequency ω and the resonance frequency ω_M. Noting that the function has strong resonances at ω, approximately at ω_M, we can approximate the admittance by

$$\frac{v(x_s)}{f_o} \simeq \frac{1}{\rho_m h A_p} \sum_M \psi_M^2(x_s) i\omega_M \frac{1}{2\omega_M[\omega-\omega_M+i\omega_M\eta/2]}$$

$$\simeq \frac{1}{2\rho_m h A_p} \sum_M \psi_M^2(x_s) \frac{\omega_M\eta/2+i(\omega-\omega_M)}{(\omega-\omega_M)^2+\omega_M^2\eta^2/4} \qquad \text{(IV.7.13)}$$

$$\equiv G + iB \quad .$$

The real part of Eq. IV.7.13 is called the mechanical conductance and the imaginary part is the susceptance. Both are fluctuating functions of frequency caused by a series of modal resonances. If we average these functions over a band of frequencies, and over locations of the excitation, we get expressions corresponding to the average real and reactive power input to the structure for a noise source operating over that band. The average value of the conductance is

$$\overline{G} = \frac{1}{2M_p} \sum_M \langle\psi_M^2\rangle_{x_s} \frac{1}{\Delta} \int_{\omega_\ell/\omega_M}^{(\omega_\ell+\Delta)/\omega_M} \frac{\eta}{2} \frac{d(\omega/\omega_M)}{\left(\frac{\omega-\omega_M}{\omega_M}\right)^2 + \frac{\eta^2}{4}} \, . \qquad \text{(IV.7.14)}$$

The integral in Eq. IV.7.14 has a large value when ω_M is included in the band and a very small value when it is not. The average number of contributions is $n_s\Delta$ where n_s is the average modal density computed above. When ω_M is in the interval, the value of the integral is approximately π. The average conductance is, therefore,

$$\overline{G} = \frac{\pi}{2} \frac{n_s}{M_p} \quad , \qquad \text{(IV.7.15)}$$

since the mean value of ψ_M^2 is unity by its normalization.

Note that the average value of the susceptance function will vanish since it is odd in $\omega-\omega_M$ and the modal density is constant. Thus, the average admittance over the band equals the average conductance given by Eq. IV.7.15. If we substitute in the expressions, the modal density found for the simply supported plate, the average conductance is

$$\overline{G} = \frac{1}{8\rho_m h\kappa c_\ell} = G_\infty \qquad \text{(IV.7.16)}$$

which is the same value as computed for the infinite plate and given in Eq. IV.7.8.

We thus have the remarkable and general result that the average value of input conductance over a band of frequencies is given by the input conductance of a similar structure, but infinitely extended. **This** result is useful because it is frequently much simplier to compute the input admittance for infinite systems than it is for finite ones. If we are, therefore, content with average values of input power over a band of frequencies rather than the detailed variations from one frequency to the next, we can get the results desired by a simple calculation.

Energy of a Resonator Attached to a Plate

As an example of how energy can be shared between vibrating systems we consider the simple linear resonator attached to a very large thin plate at the point $\underline{x_s}$ as shown in Fig. IV.13. The resonator consists of a stiffness K, a mass M and a dash-**pot** R The upper end of the dash-pot is attached to an inertial frame. We assume that a diffuse reverberant vibrational field exists on the plate and produces a transverse velocity v. The velocity at the attachment point is v_s, and the velocity of the mass is v_M. If the velocities v_s and v_M are different, there will be a net compression of the spring and a force acting to accelerate the mass and compress the dash-pot. The equation of motion is

$$f = K\int (v_s - v_M)\,dt = M\frac{dv_M}{dt} + Rv_M \quad . \qquad \text{(IV.7.17)}$$

The velocity v_s is the velocity which would exist on the plate v in the absence of the spring, less the velocity $v_r = f\overline{G}$ caused by the reaction force, the force times the admittance of the plate. We assume in using $\overline{G}$ that the plate is sufficiently large so that several modes of the plate are excited within the resonant bandwidth of the attached oscillator. The oscillator "takes an average" over several modes of the plate response when it pushes on the plate.

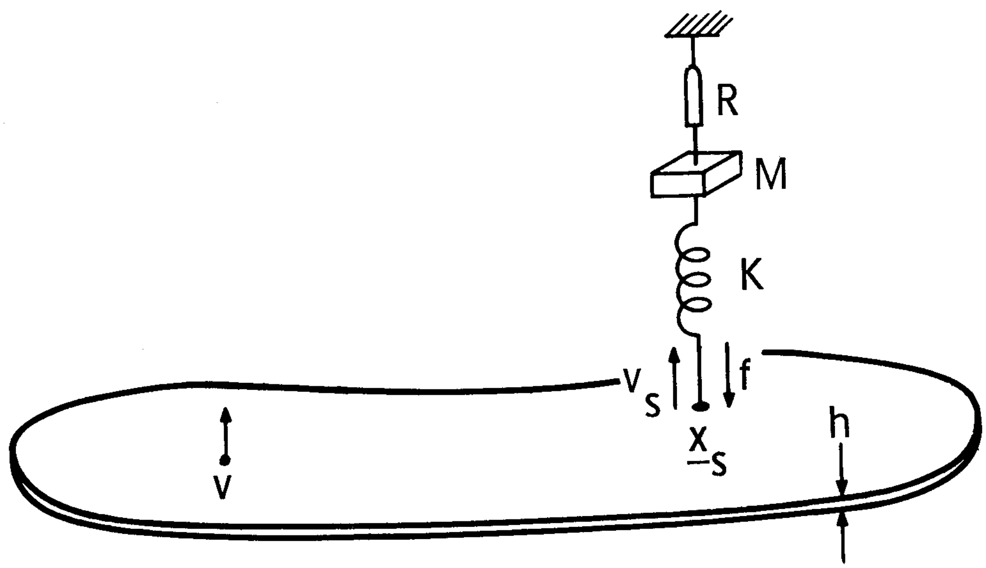

Figure IV.13.- Diagram of plate and attached resonator.

Differentiating Eq. IV.7.17 with respect to time one obtains

$$M \frac{d^2v_M}{dt^2} + R \frac{dv_M}{dt} + Kv_M = Kv_s = K(v-v_r) \quad . \qquad (IV.7.18)$$

Collecting terms, the right-hand side of this expression is

$$K(v-v_r) = K(v-\overline{G}f) = KV - k\overline{G}(M \frac{dv_M}{dt} + Rv_M) \qquad (IV.7.19)$$

which upon substituting in Eq. IV.7.18 yields

$$\frac{d^2v_M}{dt^2} + \omega_o(\eta_o+\eta_{coup}) \frac{dv_M}{dt} + \omega_o^2(1+\eta_{coup}\eta_o)v_M = \omega_o^2 v \quad , \qquad (IV.7.20)$$

where $\eta_o \equiv R/\omega_o M$ and $\eta_{coup} \equiv \omega_o \overline{G}M$. In this equation the unloaded response v or equivalently, the response at a position removed from the attachment point, acts as a source for the oscillator velocity v_M. If v has a uniform spectrum over a bandwidth Δ which encompasses the resonance ω_o, the mean square response is given by

$$\langle v_M^2 \rangle_t = \frac{\pi}{2} \frac{\omega_o}{\eta_o+\eta_{coup}} \frac{\langle v^2 \rangle_t}{\Delta} \quad . \qquad (IV.7.21)$$

If the resonator has no internal damping, then $\eta_o \rightarrow o$ and one can rewrite Eq. IV.7.21 to give

$$M\langle v_M^2 \rangle_t = \frac{M_p \langle v^2 \rangle_t}{n_s \Delta} \qquad (IV.7.22)$$

where we have used the value of $\overline{G}$ given in Eq. IV.7.15. Equation IV.7.22 is a statement of the equality of the energy of the resonator to the average energy of the modes in the plates. This "thermal equilibrium" result is common where undamped structures or sound fields are in energy contact with other systems having energy diffused uniformly in their modes of vibration. When a finite amount of damping is present in the resonator, then $\eta_o \neq 0$ and one has instead of Eq. IV.7.22

$$M\langle v_M^2\rangle_t = \frac{M_p\langle v^2\rangle_t}{n_s\Delta}\,\frac{\eta_{coup}}{\eta_o+\eta_{coup}} \quad . \tag{IV.7.23}$$

From Eq. IV.7.23 we note that the internal damping of the resonator must be comparable to its damping caused by coupling with the structure in order to effectively reduce its vibration in this environment. This is an important result in trying to establish damping limits for structures which are attached to other structures in a random environment. For example, a truss in a missile or a space craft may be very difficult to damp if it is strongly attached to the exterior structure and has a lot of coupling damping to the skin and frame members. The result also tells us that for no damping at all the maximum energy of the attached structures is limited to the average modal energy of the structure to which it is attached.

V. COUPLING BETWEEN VIBRATION AND SOUND WAVES

The two chapters immediately preceding have been principally concerned with the propagation of oscillatory disturbances in fluids and structures and with the natural resonances of finite regions. Now, we focus on interactions between structures and fluid: specifically the sound radiated from a vibrating structure and the vibration generated by incident sound waves. The type of problem we consider involves a finite structure isolated in a sound-bearing fluid. However, the modification to a structure in a large but finite room will also be presented.

The method of analysis for the structure's response to sound starts by representing the general response as a superposition of the responses of individual natural modes. In many cases of practical importance, the response of any one mode is nearly independent of the response of others; we idealize the situation by ignoring the small coupling between them. Then, the motion in each mode is analytically equivalent to the response of a suitably defined simple resonator; the motion of the whole structure is equivalent to the response of a set of resonators. These equivalent situations were studied in Chapter II.

The central analytical problems are two-fold. First, one must find the parameters (mass, stiffness, resistance) of the resonator equivalent to a mode. Second, one must find the driving force which is equivalent to a specified sound field. The techniques for solving the problems will first be formulated in general terms. We shall be concerned with developing a language of concepts for describing the interactions of sound waves and _any_ structure. Specific applications will follow.

V.1 The Equivalent Resonator for a Structural Mode

We consider the vibrations of a finite structure isolated in a sound-bearing fluid. Vibration in vacuum is a classical problem, conveniently analyzed in terms of the natural modes of the structure; only a few complications are added by the presence of the fluid.* The general vibration can be expressed as an infinite sum of terms, each of which has a different, characteristic spatial distribution of vibrational amplitude. Thus, the instantaneous vector velocity of a point, whose rest position is $\underline{r}$, is expressed by the series

*Analysis in terms of natural modes is discussed at length in most texts on vibration theory. For example, see K. N. Tong, _Mechanical Vibration_ (John Wiley and Sons, New York, 1960), Chapter 4.

$$\underline{v}(\underline{r},t) = \Sigma\ v_m(t)\underline{\psi}_m(\underline{r}) \quad , \tag{V.1.1}$$

where each term is a different mode of vibration. The scalar functions of time v_m are the modal velocities. The vector functions of position $\underline{\psi}_m$ are modal shape functions.

The natural modes for vibration in a vacuum are characterized by the vanishing of cross-product terms (involving $v_n v_m$, with $n \neq m$) in the series expansions for both kinetic and potential energy. Each expansion is expressible as a single series of terms in $|\underline{\psi}_m|^2$ if natural mode shapes are used in Eq. V.1.1.

Thus, the instantaneous kinetic energy for the vibration specified in Eq. V.1.1 is

$$T = \frac{1}{2}\int_W \rho(\underline{r})\underline{v}\cdot\underline{v}\ d\underline{r} = \Sigma_m \frac{1}{2}\ v_m^2(t) \int_W \rho(\underline{r})|\underline{\psi}_m|^2 d\underline{r} \ , \tag{V.1.2}$$

where ρ is the density and the integrals extend over the whole volume of the structure, denoted by W.

The shape functions of the natural modes are unique except for arbitrary, constant scale factors. In this study, we shall require that each $\underline{\psi}_m$ be so scaled that the integrals in Eq. V.1.2 all equal the mass of the structure:

$$\int_W \rho|\underline{\psi}_m|^2 d\underline{r} = \int_W \rho\ \ d\underline{r} = M_o \quad . \tag{V.1.3}$$

Then, the kinetic energy assumes the simple form

$$T = \Sigma_m \frac{1}{2} M_o v_m^2(t) \quad . \tag{V.1.4}$$

In the absence of internal dissipation of energy, each natural mode will resonate at its own characteristic or natural frequency, ω_m rad/sec. The value of the modal velocity $v_m(t)$, in both natural and forced vibration, is governed by the same laws as the velocity of an undamped simple resonator with mass M_o and modal stiffness

$$K_m = \omega_m^2\ M_o \quad . \tag{V.1.5}$$

The presence of internal dissipation adds two features. The energy of vibration in any one mode will be dissipated, as in the damped simple resonator. But, in general, each natural mode will also be coupled by the dissipative forces to every other mode. This latter complication is analytically undesirable, but it can be neglected on the assumption that, for small damping, the coupling forces due to dissipation are small compared with other forces. We assume, therefore, that damping is small and coupling negligible, so that the total power dissipated in a general vibration, Eq. V.1.1 is expressible as a series

$$\Pi = \Sigma R_{m,int} v_m^2(t) \tag{V.1.6}$$

where the coefficients $R_{m,int}$ are called the <u>modal resistances</u>. The subscript <u>int</u> indicates that energy dissipation is <u>internal</u> to the structure.

The coefficients M_o, $R_{m,int}$, K_m defined by these equations specify a set of uncoupled simple resonators, one for each natural mode. Each resonator is equivalent to the corresponding mode in an energetic sense: the resonator's energy functions equal the modal contributions to the total energy functions of the structure, when the velocity of the resonator's mass equals the modal velocity $v_m(t)$. To complete the modelling of forced structural vibration by the forced vibration of a set of resonators, it will be necessary to determine the forces on the resonators which are equivalent to a specified distribution of external forces on the structure.

<u>Bending vibrations</u>: The most important class of sound-excited structural vibration involves bending motion of uniform thin panels, or of uniform beams. In pure bending vibrations, the kinetic energy is associated solely with the transverse velocity, i.e. the velocity normal to the surface of the panel. The transverse velocity is uniform in the thickness direction, but varies from point to point on the surface. The volume integral in Eq. V.1.3 can then be expressed as a surface integral

$$\int_W \rho |\underline{\psi}_m|^2 d\underline{r} = \int_S m |\underline{\psi}_m|^2 d\underline{r}$$

where m is the "surface density" (mass per unit area) and S is the panel's surface. Now, if the panel is uniform so that m is constant, Eq. V.1.3 becomes very simple. It indicates that the shape functions shall be so scaled that their average values are all unity:

$$<|\underline{\psi}_m|^2>_{\underline{r} \text{ on } S} = \frac{1}{A}\int_S |\underline{\psi}_m|^2 \, d\underline{r} = 1 \quad . \tag{V.1.7}$$

where A stands for the surface area of the panel.

With this choice of scale factors, the modal velocity $v_m(t)$ is seen to be the spatial average velocity (in a root mean square sense) for vibration in one mode:

$$v_m^2(t) = <|v_m(t)\underline{\psi}_m(r)|^2>_{\underline{r}} \quad .$$

Moreover, in a general vibration involving many natural modes, the spatial average of total velocity is given by:

$$<|\underline{v}(\underline{r},t)|^2>_{\underline{r}} = \Sigma \; v_m^2(t) \quad ,$$

by virtue of the vanishing of cross-product terms.

V.2 Equivalent Forces

The equivalent modal force $f_m(t)$ must be chosen so that the instantaneous power delivered to the resonator (rate of flux of energy) is identical with the modal component of total power delivered to the structure. In the present case of excitation by sound, the external forces are those due to acoustic pressures on the surface of the structure. Let that surface be denoted by S. Then the identity of instantaneous powers is expressed analytically by

$$f_m(t)v_m(t) = -\int_{\underline{r} \text{ on } S} p(\underline{r},t)v_m(t)\underline{n}\cdot\underline{\psi}_m(\underline{r})d\underline{r}$$

where $\underline{n}$ is the unit normal vector to the surface S, directed out of the structure and into the fluid. With the minus sign, the integrand is the product of pressure by the inward normal velocity due to one mode, and thus equals the instantaneous intensity directed into the structure; the integral is the instantaneous power. When $v_m(t)$ is cancelled from both sides, we have the fundamental equation for determining equivalent forces:

$$f_m = -\int_S p \; \underline{n} \cdot \underline{\psi}_m \; d\underline{r} \quad . \tag{V.2.1}$$

Consider two structures of the same size and shape exposed to sound generated by identical sources. The sound pressures on their surfaces will not be the same unless the structures are internally identical. If, for example, the two structures have different amounts of internal damping, the velocities on their surfaces will differ; there will be a corresponding difference in the total pressure. It is convenient to separate out this response-dependent component of pressure and to treat it separately. Thus, we express the total sound pressure as a superposition of two parts,

$$p(\underline{r},t) = p_{b\ell}(\underline{r},t) + p_{rad}(\underline{r},t) \quad , \tag{V.2.2}$$

of which the first, $p_{b\ell}$, called the "blocked" pressure, is that which would exist in the absence of structural motion. It depends only on the sound sources and the shape of the structural surface S. The second part, p_{rad}, called the "radiation" pressure, is the change in total pressure due to non-vanishing motion of the structural surface. It depends only on the response velocity $\underline{v}(\underline{r},t)$ and the shape of the surface S, and not at all upon the nature of the sources that generate the response. If in the absence of sound sources, the same velocity $\underline{v}(\underline{r},t)$ could be generated by other means (say, by mechanical action), the sound pressure p_{rad} would be observed in the fluid.

Corresponding to this decomposition of the total sound pressure, the equivalent modal forces also split into two parts:

$$f_m = f_{m,b\ell} + f_{m,rad} \quad , \tag{V.2.3}$$

of which the second, the radiation force, describes the reaction of the fluid to structural motion. Now, the radiation pressure p_{rad} at any point depends on the vibrational velocity $\underline{v}$ at all points of the surface. If the velocity is expressed as a series of modal components (Eq. V.1.1), the radiation pressure will be a series of terms, each due to a different modal velocity v_m. Correspondingly, the radiation force for any one mode (derived from p_{rad} by Eq. V.2.1) will be a series and include contributions from all the modes. In other words, the fluid reaction tends, in general, to couple together the motion of different modes.

Fortunately for simplicity in calculation, it has generally been found possible to neglect the coupling forces due to radiation pressures in analyzing structural vibration caused by sound in air. Because the density of air is so small compared with the density of structures, the radiation forces coupling different modes are small and do not affect the motion sufficiently to be important in the overall picture. We shall neglect them in this study. However, one must note that the situation can be very different for structural vibration in water. In underwater sound problems, the radiation coupling forces can be an essential and complicating feature of structural vibration.

The first term of Eq. V.2.3, the blocked force for the $m^{\underline{th}}$ mode, is that to which we shall give most attention in the rest of this study. For, if only the equivalent forces be known, we can proceed to find the response of the equivalent resonators by the procedures developed in Chapter II. The blocked force is derived from the blocked pressure by an integral in the form of Eq. V.2.1. Like the blocked pressure, it depends on characteristics of the sound source and of the surface shape of the structure. In addition, because the mode shape function appears in the integral, the blocked force depends on mode shape. Indeed, it is a measure of the closeness of match between the distribution of blocked pressure and the mode shape function.* The ratio of modal blocked force to a reference sound pressure plays a role similar to that of the transfer functions H_m which were introduced in Chapter II, Section 8.

*An analytical expression of this statement is proved by Schwarz's Inequality:

$$|f_{bl}|^2 \leq A^2\left[\langle|p_{b\ell}(\underline{r})|^2\rangle_{\underline{r}\text{ on }S}\langle|\underline{n}\cdot\underline{\psi}_m(\underline{r})|^2\rangle_{\underline{r}\text{ on }S}\right]$$

where A is the surface area. Equality obtains when the ratio of blocked pressure to $\underline{n}\cdot\underline{\psi}_m$ is constant.

V.3 Coupling Parameter

Let us consider in more detail the nature of the blocked force for a single mode and its dependence on the sound field. We first consider excitation by a pure-tone plane wave of sound. It will be found that the blocked force depends on frequency, the magnitude of pressure, and the direction of incidence of the wave. The analysis for more complicated situations, involving many directions or many frequencies, can be built up from solutions for the simple case.

For convenient analysis of pure-tone variables we use complex notation. The complex pressure of the pure-tone plane wave which is incident on the structure is

$$p_{inc}(\underline{r},t) = P_o e^{i\omega t} e^{-i\underline{k}\cdot\underline{r}} \ ; \ \underline{k} = k\underline{\Omega} \ ; \ k = |\underline{k}| \ ; \qquad (V.3.1)$$

where ω is the frequency, $\underline{k}$ the wave number, and $\underline{\Omega}$ is a unit vector in the direction of propagation. This is the sound pressure that would be observed in the absence of the structure, or at points far from it. The presence of the structure changes the pressure field, by the mechanisms of reflection, diffraction, and scattering. The changes occur even if the structure is motionless; i.e. "blocked". A simple example is the doubling of pressure amplitude that arises from the reflection of a plane wave from a large plane surface (Chapter III, Section 6). However, the blocked pressure is proportional to the amplitude of incident pressure, P_o, and so also is the blocked force. In complex notation, we have

$$p_{b\ell}(\underline{r},t) = P_{b\ell}(\underline{r}) e^{i\omega t} \propto P_o e^{i\omega t} \quad ,$$

$$f_{m,b\ell}(t) = F_{m,b\ell} e^{i\omega t} \propto P_o e^{i\omega t} \quad .$$

We define a coupling parameter to express the proportionality between blocked force and incident sound pressure:

$$\Gamma_m(\omega,\underline{\Omega}) \equiv F_{m,b\ell}/P_o = -\int_S \frac{P_{b\ell}(\underline{r})}{P_o} [\underline{n}\cdot\underline{\psi}_m(\underline{r})] d\underline{r} \quad . \qquad (V.3.2)$$

This coupling parameter is a complex number, and varies with both frequency and angle of incidence $\underline{\Omega}$. Being a ratio of force to pressure, it has the dimensions of area. However, in other respects, the coupling parameter is essentially identical with the frequency-dependent transfer functions $H_k(\omega)$ which were introduced in Chapter II, Section 8: The incident sound pressure P_o is an exciting "force" which is common to all modes; the forces on the masses of the various modal resonators, $F_{m,b\ell}$, are all proportional to P_o, but the constants of proportionality vary with frequency and differ between modes.

A simple example of the coupling parameter is afforded by the case of a small rigid piston moving in the surface of a large plane wall. The blocked sound pressure is twice the incident sound pressure, because of the reflected wave. Every point of a rigid piston moves with the same velocity; therefore, the mode shape function $\underline{\psi}$ is constant. It follows from the agreed normalization, Eq. V.1.3, that $\underline{n}\cdot\underline{\psi} = 1$. Thus, the integrant in Eq. V.3.2 equals 2, and the magnitude of the coupling parameter equals twice the area of the piston, independent of both frequency and direction of incidence.

V.4 Radiation Loads

We have observed that the radiation component of sound pressure, p_{rad} in Eq. V.2.2, depends upon the response velocity of the structure and not upon the source of excitation. Again, we consider pure-tone response in a single mode. In complex notation, the response velocity is

$$\underline{v}(\underline{r},t) = V_m e^{i\omega t}\underline{\psi}_m(\underline{r}) \quad . \tag{V.4.1}$$

The radiation pressure is the reaction of the fluid to this vibration and will be proportional to V_m:

$$p_{rad}(\underline{r},t) = P_{rad}(\underline{r})e^{i\omega t} \propto V_m e^{i\omega t} \quad . \tag{V.4.2}$$

The radiation force for the $m^{\underline{th}}$ mode is, therefore, also proportional to V_m:

$$f_{m,rad}(t) = F_{m,rad}e^{i\omega t} \propto V_m e^{i\omega t} \quad .$$

The constant of proportionality between the radiation force and the modal velocity is a complex number, whose negative is called the radiation impedance

$$Z_{m,rad}(\omega) \equiv -F_{m,rad}/V_m = \int_S \frac{P_{rad}(\underline{r})}{V_m} [\underline{n}\cdot\underline{\psi}_m(\underline{r})]d\underline{r} \quad . \quad (V.4.3)$$

(The minus sign is required to convert the force F_m of the fluid on the structure into a force of the structure on the fluid.)

Several examples of the radiation impedance were presented in the earlier analysis of sound waves (Chapter III, Section 8). In the case of a large flat piston, the radiation impedance was found to be real and equal to $(\rho_o c)$ times the area of the piston. In the case of a uniformly pulsating small sphere, the impedance was found to be complex and equivalent to the mechanical combination of a small mass driven through a dashpot (Fig. III.7).

Analyses for the radiation impedance presented to vibrations of various surfaces in different modes account for a large part of the acoustical literature and are too numerous to be summarized here. However, certain common features justify engineering approximations that will considerably simplify our future calculations.

In general, the radiation impedance is complex. The imaginary part corresponds to an inertial, mass-like reaction (i.e. like ωM, where M has units of mass); however, the equivalent mass attributed to radiation loading is not constant, but varies slowly with frequency. This "virtual" or "added" mass is almost always small compared with the mass of the solid structure; the few known exceptions involve metal shell structures under water. The main effect of the added mass is to reduce the frequency of modal resonance from its value for the structure in vacuo. The reduction is generally small (less than a few percent) and negligible for structures in air.

The real part of the radiation impedance, called the radiation resistance, is usually small and often smaller than the imaginary part. However, it cannot always be neglected. Whereas the imaginary part of the radiation impedance was small compared to the corresponding inertial reaction of the structure, the radiation resistance must be compared with the structure's inherent resistance. In energetic terms, one must compare the energy dissipated in radiation with the energy dissipated internally, for the same amplitude of vibration. In such a comparison, it is often found that dissipation by radiation predominates, even for structures in air. Since the response of a structure to steady excitation builds up until energy dissipation balances the energy input, it is evident that the radiation resistance must not be neglected.

What is this energy we speak of as being dissipated in radiation, and where is it dissipated? When the structure is vibrating in one mode with velocities given in complex form as in Eq. V.4.1, the motion is impeded by the radiation pressure given in Eq. V.4.2. According to the theorem for computing a time-average by means of complex products (Eq. II.5.5), we have

$$\langle \mathrm{Re}(p_{rad})\,\mathrm{Re}(\underline{n}\cdot\underline{v})\rangle_t = \frac{1}{2}\,\mathrm{Re}\left[P_{rad}V_m^*\right]\underline{n}\cdot\underline{\psi}_m \quad .$$

This is the time average of intensity, i.e. the time average of the rate of flux of energy through the surface S, per unit area. The integral of it over the whole surface S equals the time-averaged power delivered to the fluid from the vibrating structure. According to the definition of radiation impedance, Eq. V.4.3, this time-averaged power is

$$\langle \Pi_{rad}\rangle_t = \frac{1}{2}\left|V_m\right|^2 R_{m,rad} \qquad (V.4.4)$$

$$= \langle[\mathrm{Re}(V_m e^{i\omega t})]^2\rangle_t R_{m,rad}$$

where $|V_m|^2 = V_m V_m^*$, and $R_{m,rad} = \mathrm{Re}(Z_{m,rad})$.

The energy "dissipated in radiation" is not being converted to heat. It is vibratory energy of the structure which is converted into sound waves and propagates away from the structure into the fluid. This leakage of energy from the structure can be as effective a means of diminishing its vibratory energy as is direct conversion to heat.

V.5 Modal Response Equations

In the preceding sections we have formally manipulated the equations for sound-excited structural vibration into a simple form similar to that governing the response of a simple resonator. We considered the response to a pure-tone plane wave whose complex sound pressure amplitude is P_o. The response velocity was described as a superposition of modal responses (Eq. V.1.1). Each mode behaves as a simple resonator with mass M_o, stiffness K_m, and resistance $R_{m,int}$. The equivalent modal force was separated into a blocked force proportional to P_o (Eq. V.3.2) and a radiation reaction proportional to the modal velocity V_m. Thus, the equation governing the pure-tone response of any single mode can be written

$$F_{m,b\ell} = \Gamma_m P_o = Z_m V_m \tag{V.5.1}$$

where

$$Z_m = Z_{m,rad} + Z_{m,int};$$

$Z_{m,int} = R_{m,int} + i(\omega M_o - K_m/\omega)$. The term $Z_{m,int}$ is the impedance of the simple resonator that represents one natural mode of the structure in vacuo (Eq. II.6.3). In deriving these relations we have explicitly neglected small forces, arising from internal dissipation and from sound radiation pressures, that could tend to couple the response of one mode to the response of others.

Finally, we noted that the radiation impedance $Z_{m,rad}$ is generally found to be small enough that its imaginary part can be neglected, at least for structures in air. The relative importance of the real part, the radiation resistance $R_{m,rad}$, varies from case to case. As a measure of its importance, we define a resistance ratio

$$\mu_m \equiv R_{m,rad}/(R_{m,rad} + R_{m,int}) \quad , \tag{V.5.2}$$

whose value always lies between 0 and 1.

The net result is an equation of the same type as that governing a simple mechanical resonator. However, the total resistance is the sum of radiation and internal components.

V.6 Directivity: Reciprocity

We have seen that sound energy is radiated from a vibrating structure, the total power being determined by the radiation resistance. In general, the energy flows out in all directions, but the sound intensity varies with direction. This selectivity with respect to direction is called directivity. However, a vibrating structure is also selective with respect to direction in another sense. The blocked force generated by an incident sound wave can vary with the direction of incidence; that is, the modal coupling parameter Γ_m is a function of direction. Thus, the structure exhibits directivity both when it is driven by sound and when its vibration generated sound. It is an interesting and important consequence of the principle of reciprocity that these two aspects of directivity are identical.*

Let us consider the two reciprocal situations in more detail. In the first, we assume that the structure moves with pure-tone vibration in a single mode. The complex amplitude of modal velocity is V_m. We determine the directionality of radiated sound by making pressure measurements at many points, all at the same large distance R_o from the structure but in different directions (specified by the unit vector $\underline{\Omega}$). The complex amplitude of measured pressure P_{rad} is proportional to V_m, but the factor of proportionality varies with direction $\underline{\Omega}$.

In the second situation, we evaluate the modal blocked force which results from a sound source located at these same points, all at the same large distance R_o from the structure. No matter what the direction of the source, the incident sound wave is approximately plane because the distance is large, and it has the same complex pressure amplitude P_o because the distance is the same in each case. Thus the modal blocked force is (cf. Eq. V.3.2)

$$F_{m,b\ell} = \Gamma_m(\omega,\underline{\Omega})\ P_o \quad .$$

*A detailed proof of the relations given in this section is presented in Section IV of a paper by P. W. Smith, Jr., "Response and Radiation of Structural Modes Excited by Sound," J. Acoust. Soc. Am. 34, 640-647 (1962). The proof is based on the validity of point-to-point reciprocity in the fluid.

The conclusion drawn from reciprocity is that the variations with direction $\underline{\Omega}$ of P_{rad} (in the first situation) and of $F_{m,b\ell}$ (in the second) are identical. Analytically, we find that the magnitudes of each are related as follows:

$$|P_{rad}(\underline{\Omega})/V_m| = (\rho_o \omega / 4\pi R_o)|\Gamma_m(\omega,\underline{\Omega})| \quad . \tag{V.6.1}$$

Note that the result is not explicitly dependent on the characteristics of the structure or of the mode shape. If any mode radiates strongly in a given direction, it is also strongly excited by waves incident from the same direction.

Equation V.6.1 makes it possible to relate the modal radiation resistance to average characteristics of the coupling parameter. We observed in Section 4, above, that the energy dissipated in the radiation resistance is energy that is converted to sound and propagates away from the structure. The dissipated power equals the power carried by the waves with pressure amplitude P_{rad}. The radiated sound wave at large distances from the structure is approximately a plane wave; its time-averaged intensity (power per unit area) is given by the plane-wave relationship (Eq. III.5.5b):

$$\langle I_{rad}\rangle_t = |P_{rad}|^2/2\rho_o c \quad .$$

The total radiated power is got by integrating the intensity over a spherical surface of radius R_o:

$$\langle \Pi_{rad}\rangle_t = \int_{\text{all } \underline{\Omega}} \langle I_{rad}\rangle_t \; R_o^2 d\underline{\Omega} \quad ,$$

where $d\underline{\Omega}$ is the differential of solid angle and $R_o^2 d\underline{\Omega}$ is the differential of area on the spherical surface.

The previous three equations relate the total radiated power to an integral of $|\Gamma_m|^2$. But Eq. V.4.4 expresses the same power in terms of the radiation resistance. When all are combined, one obtains*

*In integral notation, the average in angle is written

$$\langle G\rangle_{\text{all } \underline{\Omega}} = \frac{1}{4\pi}\int G \; d\underline{\Omega} \quad .$$

$$R_{m,rad} = \frac{\rho_o ck^2}{4\pi} \langle |\Gamma_m(\omega,\underline{\Omega})|^2 \rangle_{\text{all } \underline{\Omega}} \quad . \tag{V.6.2}$$

This important consequence of reciprocity is independent of the characteristics of the structure or of the mode shape. It indicates that the average (mean-square) coupling parameter for all directions of incidence cannot be large unless the modal radiation resistance is large, and vice versa.

Directivity Function: The fluctuations of $|\Gamma_m|^2$ from its average value constitute the directivity function for the mode:

$$D_m(\omega,\underline{\Omega}) \equiv |\Gamma_m(\omega,\underline{\Omega})|^2 / \langle |\Gamma_m|^2 \rangle_{\text{all } \underline{\Omega}} \quad . \tag{V.6.3}$$

As a result of the proportionality inherent in the reciprocity relation, Eq. V.6.1, the directivity function also describes the fluctuations in angle of $|P_{rad}|^2$ and of the time-averaged intensity.

V.7 Response to Noise and Diffuse Sound Fields

The formal development of basic equations is at an end. These equations (Eq. V.5.1) describe the response of a single structural mode to a wave of single frequency incident from a single direction. With certain approximations appropriate to moderately resonant structures with small fluid loading, we found the equations to be formally identical to those for a simple resonator excited through a frequency-dependent coupling.

It is time now to use these results to get prediction formulas for the more complicated situations of practical interest. Such situations include excitation by noise, with energy distributed over a frequency band; excitation by waves incident from many different angles; and response in numerous modes simultaneously. The procedures for accomplishing this end were developed in Chapter II.

It is pompous to note that the infinitude of possible situations must contain exceptions to any simple formulas. Many of the practical cases of interest are well approximated by one of several idealizations. As far as frequency content is concerned, the ideal noise is one having a flat spectrum -- the case treated in Chapter II. In regard to direction of wave incidence, the corresponding idealization is a uniform distribution of sound intensity with respect to direction of incidence. This ideal case can be described as a uniform-distribution in angle of sound sources, each of which is uncorrelated with respect to all the others. The resulting sound is called a diffuse noise field. We noted in Chapter III that the sound in a reverberant room excited by noise is approximately a diffuse field, except at the walls, of course, where sound is incident from only one side.

A. Noise response; one mode; one direction

Assume that the incident sound pressure $p_o(t)$ is a noise incident from one direction and has a broad spectral density $S_{po}(\omega)$. As a consequence of the pure-tone relationship between incident sound pressure and blocked force (Eq. V.3.2), the spectral density of blocked force is

$$S_{f,b\ell}(\omega) = |\Gamma_m(\omega,\underline{\Omega})|^2 S_{p_o}(\omega) \tag{V.7.1}$$

where Γ_m is the modal coupling parameter (compare Eq. II.8.12).

Now, the spectral density of force will not be flat even if the spectral density of pressure is perfectly flat; the frequency variations of Γ_m will modify the spectrum. However, in the practical cases that have been analyzed, it is found that the frequency variations of Γ_m are slow compared with the variations in the response curve of the mode (the plot of modal admittance), for moderately resonant modes. In other words, the bandwidth of Γ_m is large compared with the bandwidth of the modal resonance. Therefore the spectral density of force satisfies the criteria for a broad-spectrum force.

The response of a single resonator to a broad spectrum was analyzed in Chapter II (Eq. 7.6). The energy of modal response is

$$\langle E_m \rangle_t = M_o \langle v_m^2 \rangle_t = \frac{\pi}{2} S_{p_o}(\omega_m) |\Gamma_m(\omega_m,\underline{\Omega})|^2 / R_{m,tot} \tag{V.7.2}$$

where $v_m = \mathrm{Re}[V_m \exp(i\omega t)]$, ω_m is the natural frequency of the mode, and the resistance $R_{m,tot}$ includes both internal and radiation contributions.

B. Noise response; one mode; diffuse field

Now, assume that the total incident sound pressure $p_o(t)$ is the result of a diffuse noise field, with waves incident from all directions. Consider those waves which are incident in a narrow cone of directions, subtending a solid angle $d\underline{\Omega}$ sterradians. Their contribution to the spectral density S_{p_o} is

$$dS_{p_o} = S_{p_o} d\underline{\Omega} / 4\pi \quad ,$$

since, in a diffuse field, the contribution of each direction is the same. (All directions from a point constitute a solid angle of 4π sterradians.) The corresponding contribution to the spectral density of force is got from Eq. V.7.1:

$$dS_{f,b\ell} = S_{p_o} |\Gamma_m(\underline{\Omega})|^2 d\underline{\Omega} / 4\pi \quad .$$

The total spectral density of blocked force is the integral over all directions:

$$S_{f,b\ell} = \int_{\text{all } \underline{\Omega}} dS_{f,b\ell} = S_{p_o} \langle |\Gamma_m(\underline{\Omega})|^2 \rangle_{\text{all } \underline{\Omega}} \quad . \qquad \text{(V.7.3)}$$

This analysis is valid at every frequency. It is evident that the diffuse field problem differs from that for one direction only by the introduction of the average-in-angle of $|\Gamma|^2$.

The energy of one mode's response to a diffuse noise field is, therefore,

$$\langle E_m \rangle_t = M_o \langle v_m^2 \rangle_t = \frac{\pi}{2} S_{p_o}(\omega_m) \langle |\Gamma_m(\omega_m, \underline{\Omega})|^2 \rangle_{\text{all } \underline{\Omega}} / R_{m,tot} \quad \text{(V.7.4)}$$

The mean-square coupling parameter can be eliminated in favor of the radiation resistance, by means of the relation deduced from reciprocity, Eq. V.6.2. The result is

$$\langle E_m \rangle_t = M_o \langle v_m^2 \rangle_t = \frac{2\pi^2}{\rho_o c k^2} S_{p_o}(\omega_m)\, \mu_m \quad , \qquad \text{(V.7.5)}$$

where $k = \omega_m / c$ and we have introduced the modal resistance ratio μ_m (Eq. V.6.2).

This equation demonstrates an important point. No matter what the structure, the value of the resistance ratio μ_m can only range from zero (when the internal resistance far exceeds the radiation resistance) to unity (when the internal resistance is negligible). Therefore, the energy of any mode excited by a diffuse noise field reaches a finite upper limit as its internal damping is reduced towards zero. Modes that are more strongly coupled (larger values of Γ_m) are also more strongly damped by radiation, and the two effects counteract each other.

We have noted that a diffuse noise field is generated by the reverberation of sound in a room. The value of the upper limit on energy of a structural mode (Eq. V.7.5 with $\mu_m = 1$) can be shown to equal the average energy of each single acoustical mode in the room.* Thus, Eq. V.7.5 can be given this elegant interpretation: if a dissipationless structure is exposed to noise in a reverberation room, the modes of both room and structure have the same energy. This is an example of the general physical principle called <u>equipartition of energy</u>.

Let us return to consider the nature of the differences between the result for a single direction of incidence and the present result for a diffuse field. The difference is solely a matter of the directivity of the structural mode. The ratio of the responses to a noise from one direction and to a diffuse field with the same sound pressure is equal to the directivity function for that direction (Eq. V.5.3).

C. <u>Multi-modal response to a band of noise; diffuse field</u>

As a final example, consider the response of a multi-modal structure exposed to a band of noise in a diffuse field, such as a reverberant room. The calculation scheme was described in detail in Chapter II, Section 9. Basically, the procedure is to add the energies of response of all modes whose resonance frequencies lie in the frequency band of the excitation.

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*R. H. Lyon and G. Maidanik, "Power Flow Between Linearly Coupled Oscillators," J. Acoust. Soc. Am. <u>34</u>, 623-639 (1962), Section VII. The result follows from the expressions for modal energy and modal density derived in Chapter III.

We assume that the incident sound pressure $p_o(t)$ constitutes a diffuse noise field with a uniform spectral density S_{p_o} limited to a frequency band $\Delta\omega$. In a laboratory situation, there may actually be no noise with other frequencies. In a field situation, the measurement apparatus must include electrical filters; one measures the noise components in the filter bandwidth $\Delta\omega$ and predicts the response components in the same band. The two situations are essentially identical. The mean-square pressure in the band is:

$$\langle p_o^2 \rangle_t = \int_{\Delta\omega} S_{p_o} d\omega = \Delta\omega \; S_{p_o} \quad . \tag{V.7.6}$$

The total energy of response is, approximately, the sum of the modal energies of modes with ω_m in the band. The sum, in turn, equals the product of the number of modes by the average energy per mode. The number of modes equals the modal density $n(\omega)$ times the bandwidth $\Delta\omega$. Analytically, the time-averaged total energy is

$$\langle E \rangle_t = \Sigma_m \langle E_m \rangle_t = n(\omega) \; \Delta\omega \; \langle E_m \rangle_{t,m} \quad . \tag{V.7.7}$$

The average energy per mode can be computed by averaging whichever of the previous equations is appropriate and convenient. For example, using Eqs. V.7.4 and V.7.5, one finds the total response energy in the band to be

$$\langle E \rangle_t = \langle p_o^2 \rangle_t \frac{\pi}{2} n(\omega) \left\langle \langle |\Gamma_m|^2 \rangle_{\underline{\Omega}} / R_{m,tot} \right\rangle_m \quad . \tag{V.7.8}$$

Further development of the relationship depends on circumstances. If, for example, internal damping is known to predominate and the damping of every mode is about the same, then $R_{m,tot}$ is a constant. Equation V.7.8 becomes

$$\frac{\langle E \rangle_t}{\langle p_o^2 \rangle_t} = \frac{\pi}{2} \frac{n(\omega) \langle |\Gamma_m|^2 \rangle_{\underline{\Omega},m}}{R} = \frac{\pi}{2} \frac{n(\omega) \langle |\Gamma_m|^2 \rangle_{\underline{\Omega},m}}{\eta \omega M_o} \quad , \tag{V.7.9}$$

where the last expression introduces the loss factor η.

Finally, we recall that, for bending vibrations of uniform plates and beams, the time-averaged energy is simply related to the space-time averages of velocity and of acceleration

$$\langle E \rangle_t = M_o \langle v^2 \rangle_{\underline{r},t} = \frac{M_o}{\omega^2} \langle a^2 \rangle_{\underline{r},t} \quad . \tag{V.7.10}$$

(See Eq. V.1.7 and Eq. II.7.7.)

D. Example

The face of a spring-mounted piston lies flush in the side of a large box exposed in air to a diffuse, broad-band noise field (Fig. V.1). We shall compute the response, using cgs units. The parameters of the piston resonator are:

mass M	= 200 g
face area, A	= 30 cm^2
resonance frequency, ω_o	= 1000 π rad/sec (500 c/s)
loss factor, η	= 5 x 10^{-3}.

The loss factor has been determined from measurements of the reverberation time, or of the decay rate of natural vibrations (Chapter II, Section 4).

First, we must determine the coupling parameter $|\Gamma|$ for different directions of wave incidence. The piston is small compared with the wavelength (about 70 cm). Therefore, the pressure is essentially uniform across its face, and the force on the piston equals its face area times the pressure. The box is large. Therefore, the piston is shielded from waves incident from the opposite side; such waves generate little blocked pressure. On the other hand, waves incident directly upon the face containing the piston will generate a blocked pressure equal to twice the incident pressure (the pressure-doubling effect of reflection from a large wall).* It follows directly from these considerations (cf. Eq. V.3.2) that

$$|\Gamma| \approx \left\{ \begin{array}{ll} 2A, & \text{direct incidence on piston} \\ 0, & \text{shielded from piston} \end{array} \right\} \quad .$$

*In the sense used in this paragraph, a dimension L is small if 1/2 kL $\lesssim$ 1 (L $\lesssim$ 1/3 wavelength) and large if 1/2 kL $\gtrsim$ 2. See T. Nimûra and Y. Watanabe, J. Acoust. Soc. Am. 25, 76-80 (1953).

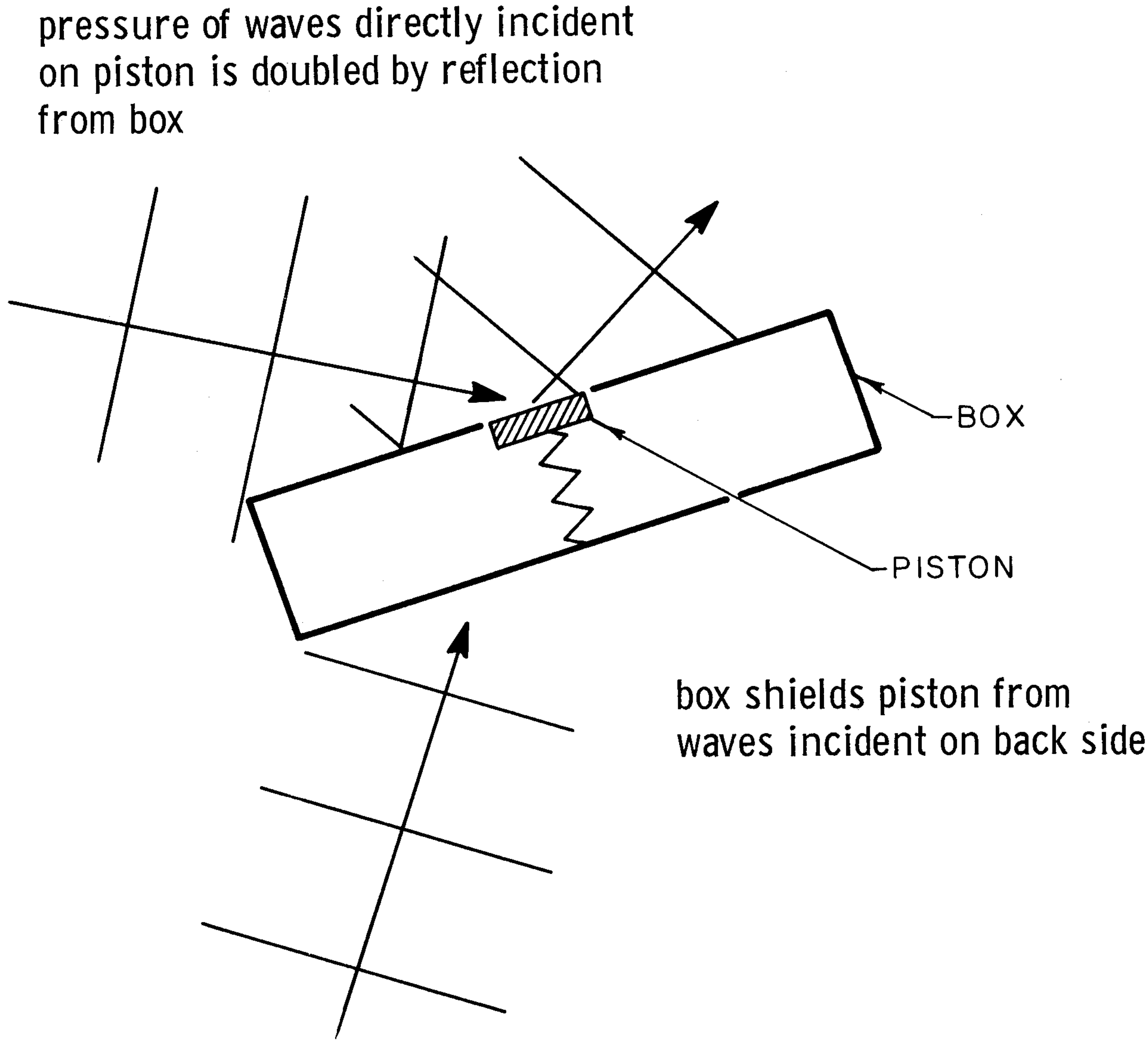

Figure V.1.- Spring-mounted piston in large box in diffuse sound field.

But, half of the waves are shielded and half are directly incident; therefore

$$\langle|\Gamma|^2\rangle_{\text{all } \underline{\Omega}} \approx 2A^2 = 1.8 \times 10^3 \text{cm}^4 \quad .$$

Note the values of the directivity function Eq. V.6.3:

$$D(\underline{\Omega}) \approx \begin{Bmatrix} 2, \text{ direct incidence on piston} \\ 0, \text{ shielded from piston} \end{Bmatrix}$$

The radiation resistance can now be computed from Eq. V.6.2. (For air at normal temperature and pressure, $\rho_o c = 42$ dyne sec/cm^3; $c = 3.45 \times 10^4$ cm/sec; $k \equiv \omega/c$.) The result is

$$R_{rad} = \frac{\rho_o ck^2}{4\pi} \langle|\Gamma|^2\rangle = 50 \text{ dyne sec/cm} \quad .$$

For comparison with the measured loss factor, we compute the "radiation" loss factor, i.e.

$$\eta_{rad} \equiv R_{rad}/\omega_o M = 0.80 \times 10^{-4} \quad .$$

Evidently, internal damping greatly exceeds damping by sound radiation; the resistance ratio (Eq. V.5.2) is

$$\mu \equiv R_{rad}/R_{tot} = \eta_{rad}/\eta = 1.60 \times 10^{-2} \quad .$$

The sound pressure is measured, at a point not near the box but exposed to waves incident from all directions. By means of electrical filtering, the measurement is restricted to frequency components in the octave band from 300 to 600 c/s. The measured sound pressure level* is 130 dB re 2×10^{-4} dyne/cm^2.

*The sound pressure level (in decibels) corresponding to a pressure p is defined as

$$10 \log_{10}(p/p_{ref})^2$$

where the standard reference pressure is 2×10^{-4} dyne/cm^2.

Thus, the mean-square pressure for the band is

$$\langle p_o^2 \rangle = (2 \times 10^{-4})^2 \times 10^{13} = 4 \times 10^5 \text{ dyne}^2/\text{cm}^4 \quad .$$

The average spectral density in the band is

$$S_{p_o} = \langle p_o^2 \rangle / \Delta\omega = 4 \times 10^5 / 600\pi = 2.1 \times 10^2 \; \frac{\text{dyne}^2/\text{cm}^4}{\text{rad/sec}} \quad .$$

The mean-square response velocity can be computed from Eq. V.7.5 or from Eq. V.7.4 (with $R_{tot} = \eta\omega_o M$):

$$\langle v^2 \rangle_t = \frac{1/2 \; \pi \; S_{p_o} \langle |\Gamma|^2 \rangle}{\eta\omega_o M^2} = 0.95 \text{ cm}^2/\text{sec}^2 \quad .$$

The rms values of velocity, displacement, and acceleration are respectively:

$$\langle v^2 \rangle^{1/2} = 0.97 \text{ cm/sec}$$

$$\langle x^2 \rangle^{1/2} = \langle v^2 \rangle^{1/2} / \omega_o = 3.1 \times 10^{-4} \text{cm}$$

$$\langle a^2 \rangle^{1/2} = \omega_o \langle v^2 \rangle^{1/2} = 3.1 \times 10^3 \text{ cm/sec}^2$$

The last is 3.1 times the gravitation acceleration.

If the same sound pressure is generated by a wave incident from a single direction, the mean-square response is modified by a factor equal to the directivity function. In the present case, $D(\Omega) = 2$ for all angles of direct incidence on the piston. Thus, for any directly incident wave or any combination of them having the same sound pressure level, the mean-square response is twice that calculated for the diffuse field, and the rms response is larger by $\sqrt{2}$.

VI. RESPONSE OF SUPPORTED PANELS TO A SOUND FIELD

As an application of the procedures and methods of computing response described in Chapter V, we shall consider the modal response of thin, flat panels to an incident sound field of broad band noise. Our development will proceed in three steps: a review of the pertinent response parameters as derived in Chapter V, a summary of the vibrational mode behavior of simply supported panels, and a calculation of modal radiation and response behavior. By considering the average behavior of a group of modes in some frequency band, we are able to predict the response of a supported panel as a combination of the responses of all affected modes in that frequency band.

VI.1 Response and Radiation Parameters

In Eq. V.7.8 the average energy of a group of modes was presented in terms of the rms pressure, the modal density, the total modal damping, and an average of the squares of the coupling factor modulus $|\Gamma|$ over the angular distribution of incident acoustic energy. When the sound is completely diffused, the average over coupling parameters can be related to the modal radiation resistance according to Eq. V.6.2. If all the sound is incident on the panel from a particular direction, then the average over direction introduces the directivity factor $D(\Omega)$, defined by Eq. V.6.3. For these two cases, therefore, we can obtain the important response parameters by considering how the panel modes radiate sound.

VI.2 Vibratory Modes of a Simply Supported Rectangular Panel

The radiation of sound will be treated below by considering the coupling factor statistics of simply supported rectangular panels. A brief review of the form and dynamics of panel modes is therefore appropriate.

In Fig. IV.7 we sketched the outline of a simply supported panel having dimensions $\ell_1 \times \ell_2$. The modal shape functions for the normal modes are, by Eq. IV.5.14

$$\begin{aligned}\psi_m(x) &= 2 \sin \frac{m\pi x_1}{\ell_1} \sin \frac{n\pi x_2}{\ell_2} \\ &= 2 \sin k_1 x_1 \sin k_2 x_2 \quad .\end{aligned} \tag{VI.2.1}$$

The wave number components for resonant modes were plotted in Fig. IV.8, and the average modal density of the panel was given in Eq. IV.5.19.

The important dimensions of the panel motion as far as its radiation behavior are concerned are its overall dimensions ℓ_1, ℓ_2, the "trace" wavelength of bending waves $\lambda_1 = 2\pi/k_1$, and $\lambda_2 = 2\pi/k_2$, and the acoustic wavelength λ_a. Throughout the discussion in this Chapter, we shall assume that the acoustic wavelength is smaller than the panel dimension. References to more complete discussions will be reviewed in Chapter VII. In Fig. VI.1 we have indicated how the wave number "modal lattice" may be divided into regions where

(a) the acoustic wave number k_a exceeds the bending wave number k_b; these modes will be termed "surface" modes.

(b) k_b exceeds k_a, but one or the other trace wave number is less than k_a; these modes will be termed "edge" modes.

(c) k_b exceeds k_a, and either trace wave number exceeds k_a as well; these modes will be termed "corner" modes.

When $k_a > k_b$, the bending wave speed in the panel exceeds the sound speed. Modes in category (a) are accordingly termed AF (acoustically fast), while modes in categories (b) and (c) are termed AS (acoustically slow). In flat plates, as we saw in Chapter IV, the bending wave speed varies directly with the wave number k_b, so that only modes that resonate above a certain frequency f_c given by

$$f_c = \frac{1}{2\pi} \frac{c^2}{\kappa c_\ell} \tag{VI.2.2}$$

will be acoustically fast. As we shall note in Chapter VII, we can encounter AF modes at lower frequencies than this when the stiffening effects of curvature are present.

In addition to these considerations, the radiative properties of the panel depend on the ratio of acoustic wavelength to panel size. There is generally a significantly larger radiation of sound when the panel size exceeds one-third of an acoustic wavelength. This is the condition we shall assume in all the discussions of this Chapter.

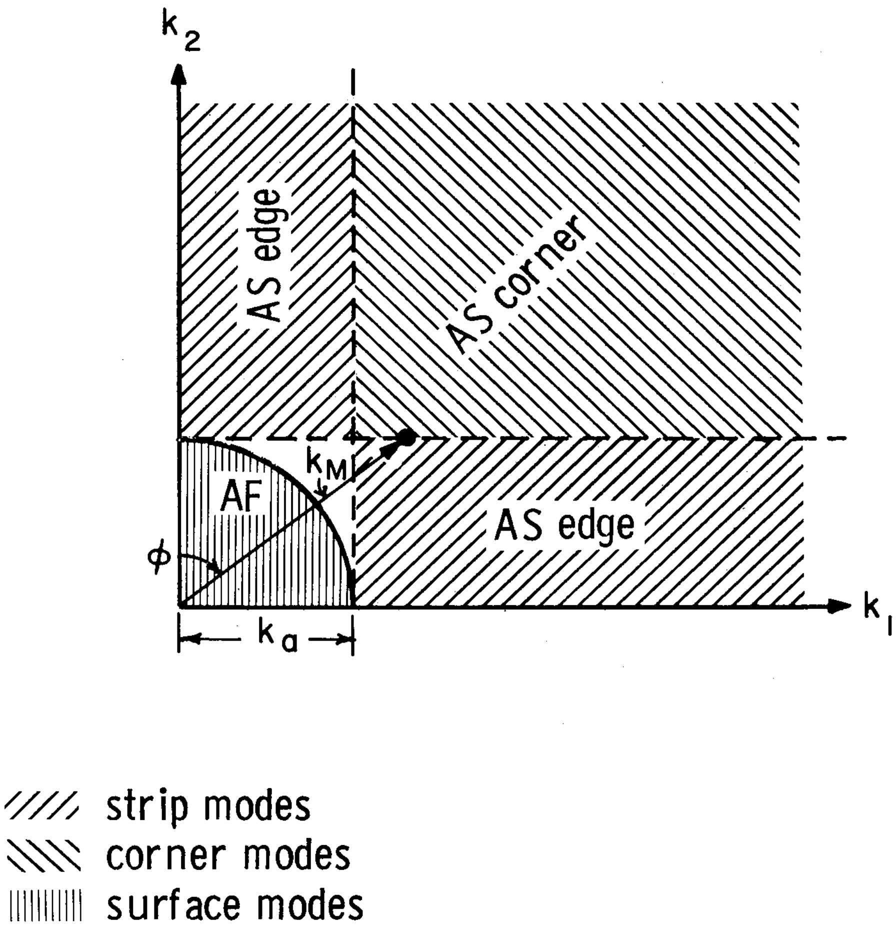

Figure VI.1.- Regions of modal radiation behavior.

VI.3 Calculation of Modal Radiation and Directivity

In this section, we shall compute the radiation resistance and directivity characteristics of the simply supported panel mounted in a large acoustic baffle of several acoustic wavelengths in size. Our procedure is basically that outlined in Chapter V.

Radiation from surface modes. When the bending wave is faster than the speed of sound and the panel is several wavelengths in size, then a relatively simple form of sound radiation results. We can evaluate $<|\Gamma|^2>_\Omega$ in this case by referring to the wave number diagram in Fig. VI.2. A diffuse sound field will produce trace wave numbers on the panel $|\underset{\sim}{k}_s| < k_a$. The relations between averages in the $\underset{\sim}{k}_s$ plane and Ω are derived in Appendix I.

The coupling factor is given by

$$\Gamma_M(\Omega) = -\int_S \frac{P_{b\ell}}{P_o} \psi_M(\underline{r})d\underline{r}$$

$$= -4\int_{A_p} e^{-ik_x x - ik_y y} \sin k_{M,x} x \sin k_{M,y} y \, dxdy$$

$$= -4A_p I_x I_y \quad , \tag{VI.3.1}$$

where $A_p = \ell_1 \ell_2$, the panel area.

Thus,

$$<|\Gamma_M(\Omega)|^2> = 16A_p^2 <|I_x|^2 \, |I_y|^2>_\Omega \quad . \tag{VI.3.2}$$

The functional forms and average values of $|I|^2$ are described in some detail in Appendix II. The various conditions listed in Section VI.2 are distinguished in the analysis by the way they affect the evaluation of $|I|^2$.

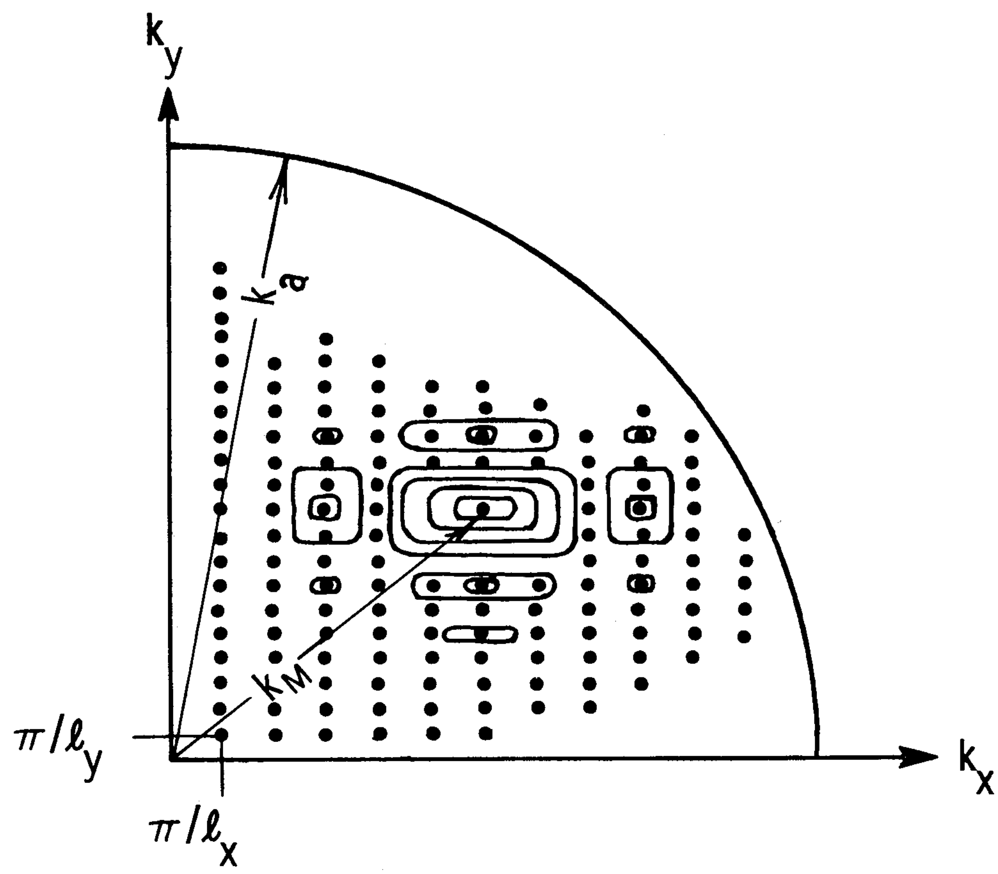

Figure VI.2.- Contours of $|\Gamma|^2$.

When $|k_M| = k_b(\omega_M) < k_a$, then referring to Fig. AII.1, the average over Ω (which is an average over $\underset{\sim}{k}_s$ according to Appendix I) has a primary maximum at $\underset{\sim}{k}_s = \underset{\sim}{k}_M$. Using Eq. AI.4, we can express Eq. VI.3.2 as

$$<|\Gamma_M(\omega)|^2>_\Omega = \frac{1}{2} <|\Gamma|^2(1-k_s^2/k_a^2)^{-1/2}>_{k_s<k_a} \tag{VI.3.3}$$

where we have related $\cos\theta$ to the acoustic wave number k_a and its trace on the panel. If we next integrate over the maximum at $\underset{\sim}{k}_s = \underset{\sim}{k}_M$, using Eq. AII.10, we get

$$<|\Gamma_M|^2> = 16A_p^2 \cdot \frac{1}{2}(\cos\theta_M)^{-1} \frac{\pi^2}{4A_p} \frac{4}{\pi k_a^2}$$

$$= \frac{8\pi A_p}{k_a^2 \cos\theta_M} \quad . \tag{VI.3.4}$$

Using Eq. V.6.2,* the radiation resistance is just

$$R_{M,rad} = \frac{A_p\rho_o c}{\cos\theta_M} \quad . \tag{VI.3.5}$$

When $k_M \lesssim 1/2\, k_a$, one can take $\cos\theta_M \sim 1$ and the radiation resistance becomes that of a large flat plate (see Eq. III.8.1) which moves uniformly normal to its rest position.

The directional characteristics of response and radiation are found from Fig. VI.2. We note that the maximum of $|\Gamma|^2$ occurs when $\underset{\sim}{k}_s = \underset{\sim}{k}_M$, and, since the magnitude of I is not affected by the sign of k_x or k_y, there will also be maxima at the reflections of k_M shown in Fig. VI.2. The four lobes of radiation maxima are sketched in Fig. VI.3. Using Eq. V.6.3, the directivity index for the radiation at the radiation maxima is

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*In studying the flat panel, we have excluded waves from one side of the panel, and the factor $(4\pi)^{-1}$ in Eq. V.6.2 should be replaced by $(2\pi)^{-1}$.

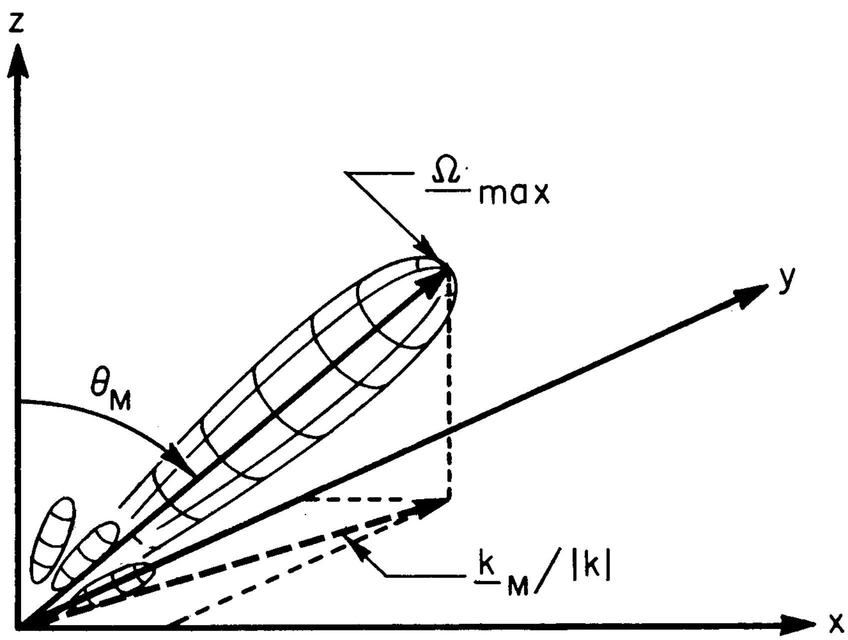

Figure VI.3.- Directivity pattern for a surface mode.

$$D_{max} = (k_a^2 A_p \cos\theta_M)/8\pi \quad . \qquad \text{(VI.3.6)}$$

The radiation resistance VI.3.5 is that of a straight crested bending wave on an infinite plate, if one considers the power radiated from an area A_p. The effective radiating area of the plate therefore is the entire surface, as indicated by the cross-hatched region in Fig. VI.4.

Radiation from edge modes. When the bending wavelength is smaller than the acoustic wavelength, but one trace wavelength, say λ_y, exceeds λ_a, a different computation from the above is required. Since we also assume that $\lambda_a<\ell_x$, then the average of $|\Gamma|^2$ can be written (see Fig VI.5)

$$<|\Gamma_M|^2>_\Omega = 16A_p^2 <|I_x|^2 \; |I_y|^2 \; (\cos\theta)^{-1}>_{k_s<k_a}$$

$$= \frac{64}{\pi k_a^2} A_p^2 \int_0^{k_a} dk_y |I_y|^2 \int_0^{k_a \sin\phi} dk_x |I_x|^2 (\cos\theta)^{-1}$$

(VI.3.7)

With the above conditions on wavelength and geometry, $|I_x|^2$ is a rapidly fluctuating function of average value $2/k_{M,x}^2\ell_x^2$ according to Eq. AII.7. Replacing $|I_x|^2$ in the integrand by its average value, and writing

$$\cos\theta = (k_a^2 \sin^2 \phi - k_x^2)^{1/2} k_a^{-1} \quad , \qquad \text{(VI.3.8)}$$

the integral over k_x can be carried out to give $\pi k_a/2$.

Since $\lambda_y>\lambda_a$, $|I_y|^2$ exhibits a strong peak which acts like a δ-function with area $\pi/2\ell_y$ according to Eq. AII.9. The value of Eq. VI.3.8 is therefore approximately given by

$$<|\Gamma_M|^2>_\Omega = \frac{64}{\pi k_a^2} A_p^2 \cdot \frac{\pi}{2\ell_y} \cdot \frac{2}{k_{M,x}^2 \ell_x^2} \cdot \frac{\pi k_a}{2}$$

$$= 32\pi \frac{\ell_y}{k_a k_{M,x}^2} \quad . \qquad \text{(VI.3.9)}$$

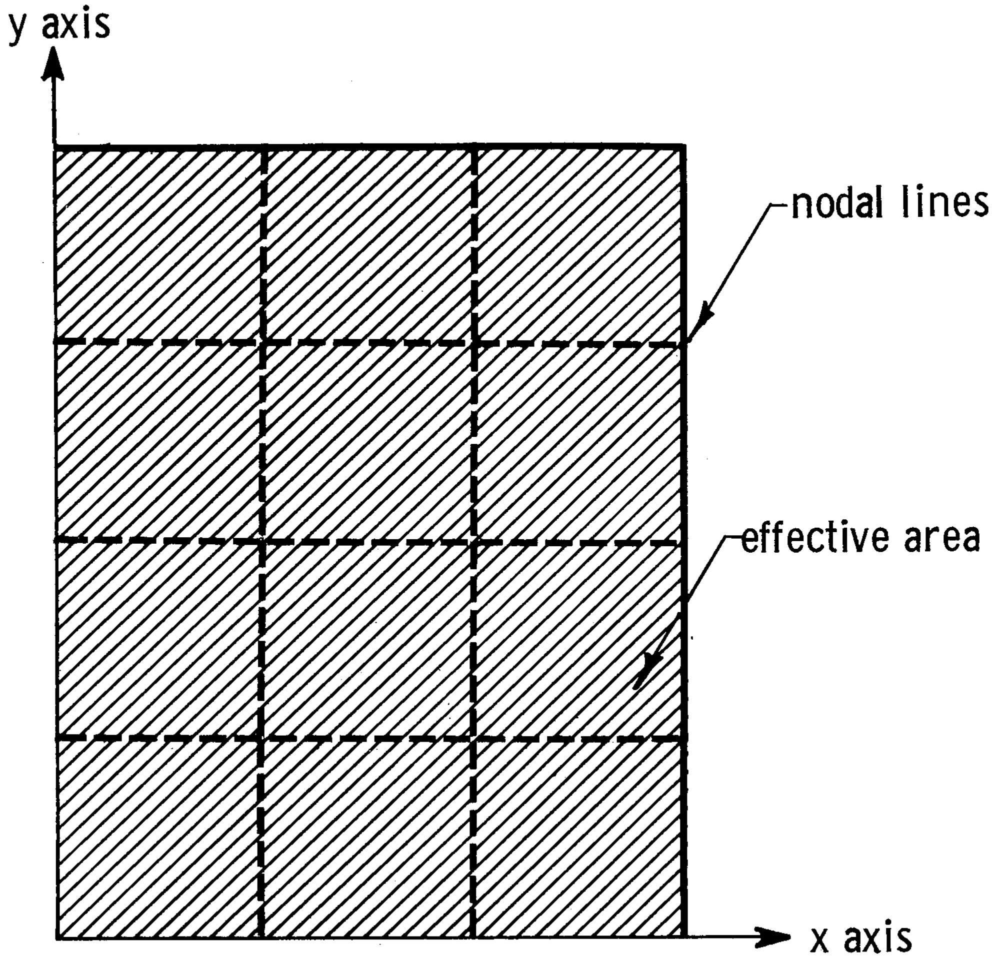

Figure VI.4.- Effective radiating area for acoustically fast (surface) mode.

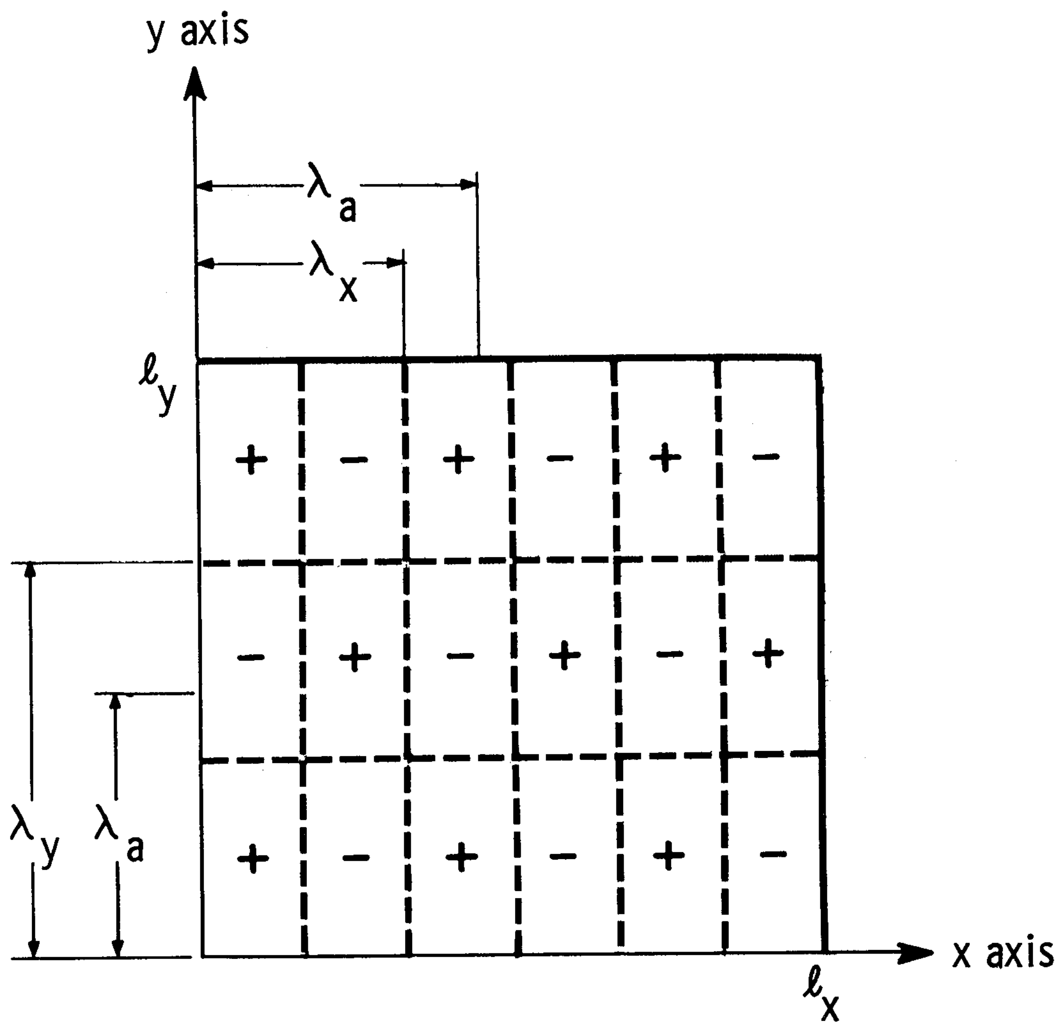

Figure VI.5.- Relations between wavelengths for acoustically slow edge mode.

If we use Eq. V.6.2 again, we get

$$R^{y}_{M,rad} = \rho_o c A_p \, 16 \frac{k_a \ell_y}{k^2_{M,x} A_p} \quad . \qquad \text{(VI.3.10)}$$

For a mode where $\lambda_x > \lambda_a$, $\lambda_y < \lambda_a$, and $\lambda_a > \ell_y$, one has the symmetric situation

$$R^{x}_{M,rad} = \rho_o c A_p \, 16 \frac{k_a \ell_x}{k^2_{M,y} A_p} \quad . \qquad \text{(VI.3.11)}$$

Another way of computing the radiation resistance is to note that volume velocity from neighboring phase cells will tend to cancel everywhere except at the edges, as shown in Fig. VI.6. When the breadth ℓ_x is greater than $\lambda_a/3$, the uncancelled volume velocity strips tend to radiate independently, and one computes the radiation resistance (VI.3.11). When $\lambda_a > \ell_x$, the strips interact. If the mode number M,x is odd, the strips will be in phase and radiation will be augmented. When M,x is even, the volume velocity is cancelled and the radiation is diminished. We should emphasize, however, that the same directivity and radiation resistance results from either the generalized force calculation above, or the combination of edge strip radiators suggested here. Details of these cancellation and enhancement effects can be found in the paper by Maidanik.*

Radiation from corner modes. When the trace wavelengths $\lambda_{M,x}$ and $\lambda_{M,y}$ are both smaller than the acoustic wavelength λ_a, then the approximation of Eq. AII.7 is applied to the integrals over both k_x and k_y. However, in this case it is simpler not to convert to an average in k_s from Ω since the integrand $|I_x|^2 |I_y|^2$ is essentially uniform with an average value

$$|I_x|^2 |I_y|^2 \simeq \frac{4}{k^2_{M,x} k^2_{M,y} A^2_p} \quad . \qquad \text{(VI.3.12)}$$

* G. Maidanik, "Response of Ribbed Panels to Reverberant Acoustic Fields," J. Acoust. Soc. Am., 34 809 (1962).

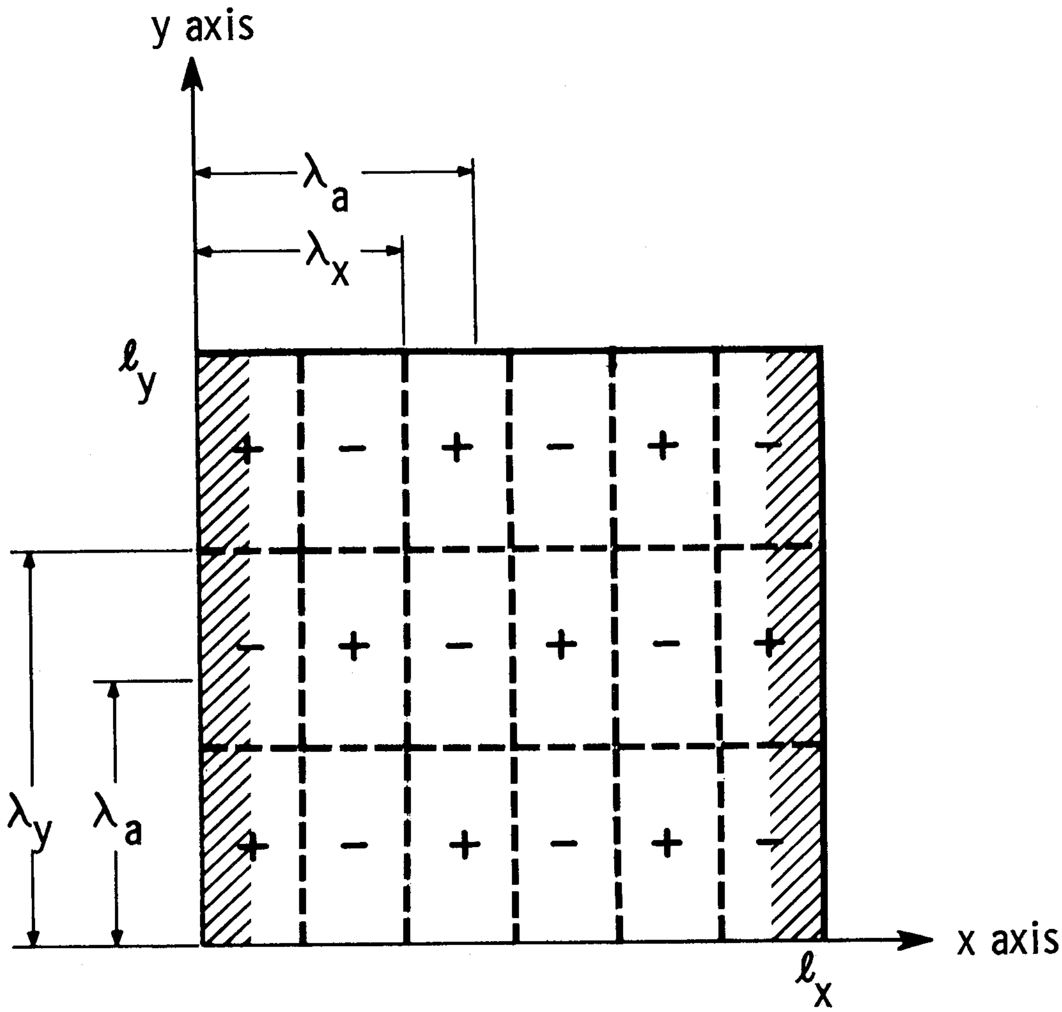

Figure VI.6.- Volume velocity cancellation for a y-edge mode.

Thus,

$$<|\Gamma|^2>_\Omega = 16A_p^2<|I_x|^2|I_y|^2>_\Omega = \frac{64}{k_{M,x}^2 k_{M,y}^2}$$

and

$$R_{M,rad} = \rho_o c A_p \frac{32}{\pi} \frac{k_a^2}{k_{M,x}^2 k_{M,y}^2 A_p} \qquad \text{(VI.3.13)}$$

using Eq. V.6.2 again (but averaged over 2π instead of 4π steradians).

In Fig. VI.7, we show the wavelength relationships that have been assumed in this development. Since the acoustic wavelength is greater than either trace wavelength, there is volume velocity cancellation everywhere except at the four corners, shown shaded. These corners act as point sources of sound which radiate independently as long as $\lambda_a<3\ell_x$ or $3\ell_y$. If one computes the sound radiation from these four sources radiating independently using the uncancelled volume velocity and the radiation resistance of a small source (similar to Eq. III.8.8, except that in the present case the source radiates into 2π steradians), then the calculated radiation resistance is exactly given by Eq. VI.3.13.

The peaks in directivity are, by Eq. V.6.3, associated with values of k_s that produce maxima in $|\Gamma|^2$. Since there are many such peaks, fairly uniform in amplitude for corner modes when $\lambda_a<\ell_x$, ℓ_y, the corner modes tend to be fairly omnidirectional in radiation (as well as responsiveness).

VI.4 Average Radiation and Response of the Supported Plate in Frequency Bands

In Section V.7, we indicated that the power absorbed by a set of modes in a diffuse random noise sound field would be determined by the average radiation resistance for the set of modes in the frequency band under consideration. In this section, we will compute the average modal radiation resistance using the results of the preceding section.

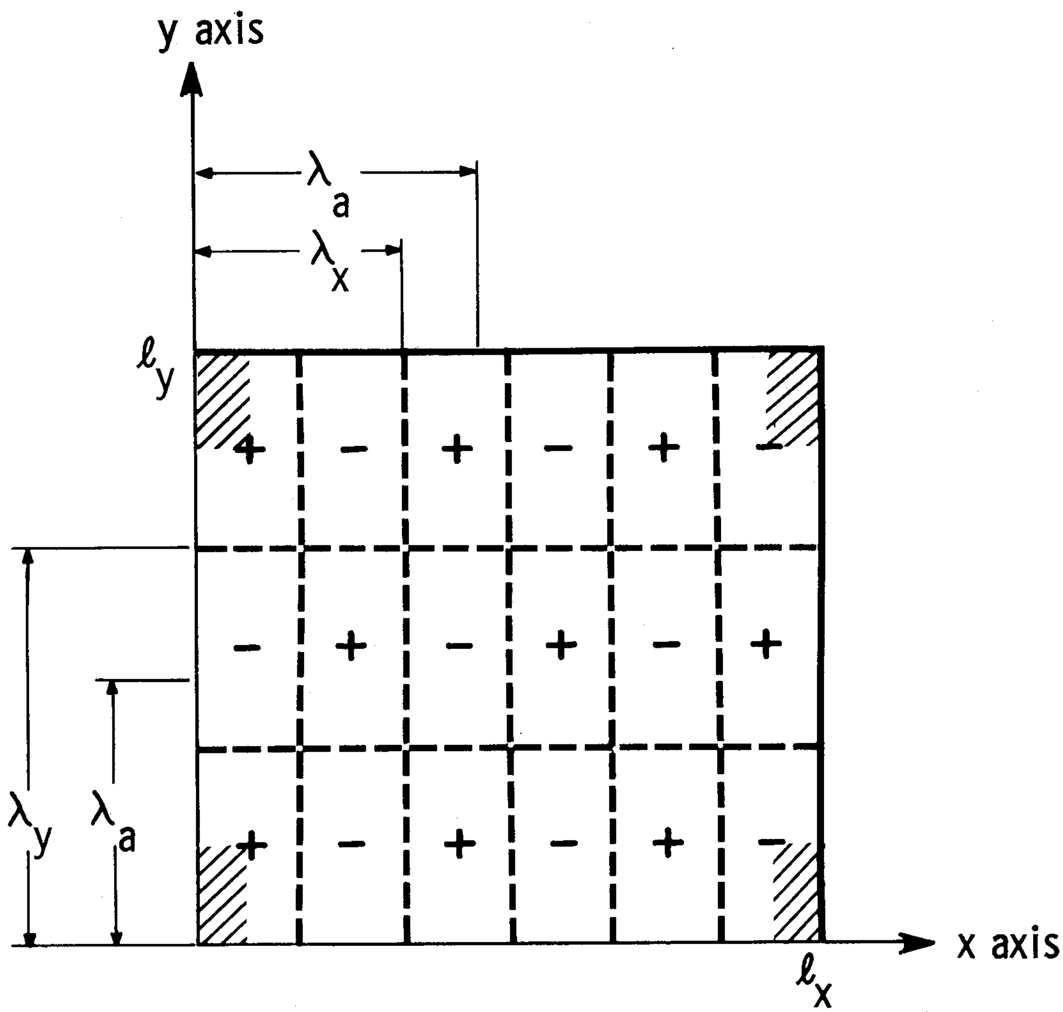

Figure VI.7.- Volume velocity cancellation for a "corner" mode.

The major divisions between classes of modes were sketched in Fig. VI.1. For $k_M<k_a$ or $\omega>\omega_c = c^2/\kappa c_\ell$, all modes are acoustically fast. Since the AF modal radiation given in Eq. VI.3.5 is independent of azimuth ϕ, we can immediately write the average radiation resistance for AF modes on a plate for which $\lambda_a<\ell_x$, ℓ_y to be

$$R_{rad}^{AF} = \frac{\rho_o c A_p}{(1-\omega_o/\omega)^{1/2}} \qquad (\omega > \omega_c) \quad . \tag{VI.4.1}$$

At frequencies less than the critical frequency, both edge and corner modes must be included in the averaging. Referring to Fig. VI.8, the relative number of y- edge modes excited by a band of noise is $(2/\pi) \sin^{-1} k_a/k_b$, and the fraction of x- edge modes is the same. The fraction of corner modes is of course $1 - (4/\pi) \sin^{-1} k_a/k_b$. Referring to Eq. VI.3.11 and Fig. VI.8, note that $k_{M,x} \simeq k_b$ in the expression for $R_{M,rad}^{y}$ and $k_{M,y} \simeq k_b$ in the expression for $R_{M,rad}^{x}$. Thus, at frequencies well below critical, the average radiation resistance for edge modes is

$$R_{rad}^{edge} = \rho_o c A_p \frac{8}{\pi^2} \frac{\lambda_c P}{A_p} (\omega/\omega_c)^{1/2} \tag{VI.4.2}$$

where $\lambda_c = 2\pi c/\omega_c$, the acoustic wavelength at the critical frequency and $P = 2(\ell_x + \ell_y)$ is the panel perimeter. More complete expressions are worked out in the reference by Maidanik.

It is possible to compute average values for the corner mode radiation loss factor but it generally turns out that the radiation from these modes is insignificant. To see this, we note that the maximum corner mode radiation resistance determined by Eq. VI.3.13 would occur for $K_{M,x} = k_b$ (or vice versa) to give

$$R_{rad}^{corner} \sim \rho_o c A_p \frac{8}{\pi^3} \frac{\lambda_c^2}{A_p} \frac{\omega_c}{\omega} \quad . \tag{VI.4.3}$$

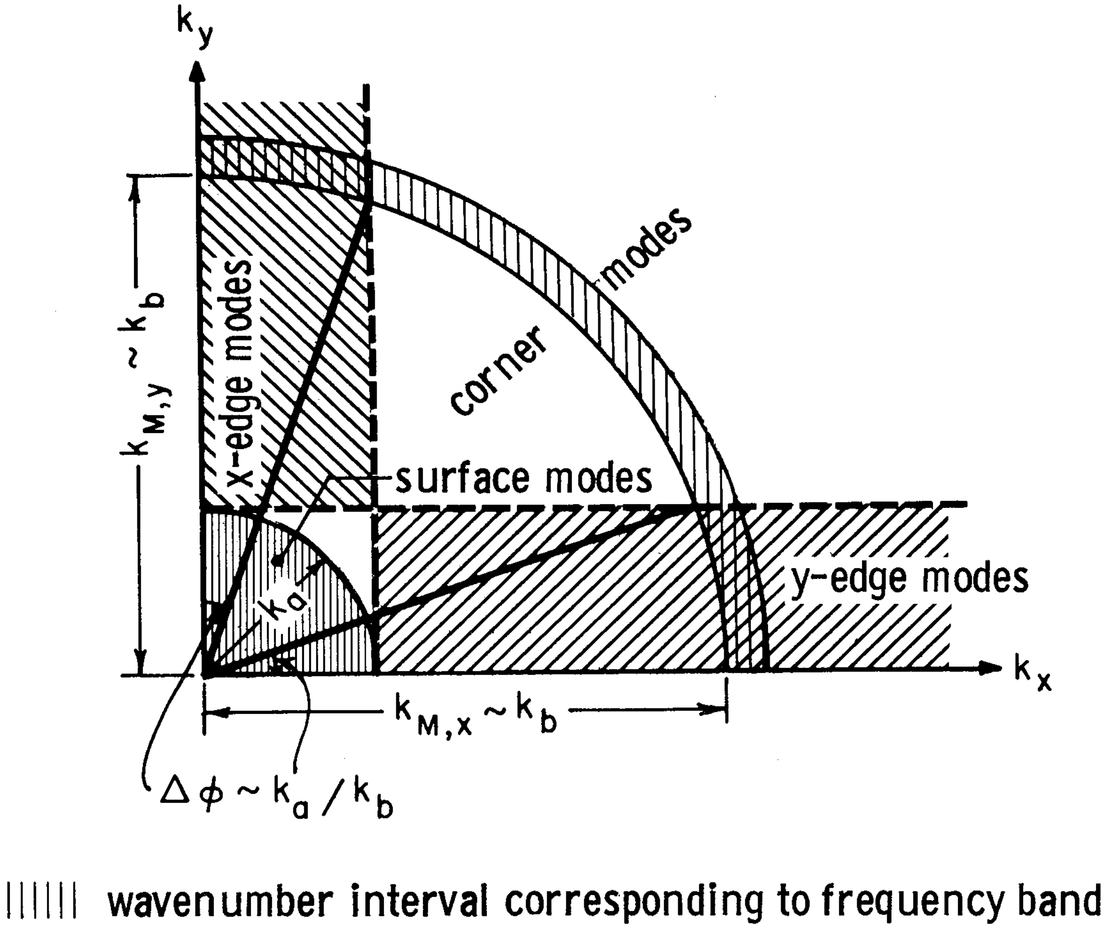

Figure VI.8.- Radiation classification of modes in k-space.

If we assign this value to all the corner modes in the frequency band (which is excessive, of course), and take the ratio of edge to corner radiation resistance, we get

$$\frac{R_{rad}^{edge}}{R_{rad}^{corner}} = \pi \frac{P}{\lambda_c} (\omega/\omega_c)^{3/2} \quad . \tag{VI.4.4}$$

Suppose we have a panel, dimensions 1 x 2 ft. of .032 aluminum. Four octaves below the critical frequency of 16 kc, this ratio is

$$\frac{R_{rad}^{edge}}{R_{rad}^{corner}} = \frac{\pi}{64} \frac{6}{1/16} \simeq 5 \quad . \tag{VI.4.5}$$

It is clear that the edge radiation will tend to dominate in the AS region. This result tends to hold for most panels of structural interest.

VII. AN INTRODUCTION TO THE LITERATURE ON APPLICATIONS OF THE ENERGY METHOD

VII.1 Introduction

The plan for this chapter is somewhat different from those that have preceded it. Until now we have attempted to proceed with the fairly careful and pedagogical development of a method of analyzing the interaction of structures with sound fields. In Chapter VI we described the way that one calculates the energy input to a simply supported rectangular panel, using the general development which preceded it. In the present chapter we shall not delve into the details of the analyses, but rather attempt to guide the reader through several published papers which deal with one aspect or another of the response of structures to random environments. In most cases these developments are based fairly directly on the analyses which have been presented here, but in some cases further developments have been necessary and these will be pointed out in the course of the discussion.

The papers and analyses covered here fall in five major categories. They are: 1) the acoustic response and radiation of flat panels; 2) the response and radiation of orthotropic and curved panels; 3) the effects of fluid loading on panels; 4) the response of panels to impact noise and other localized excitations such as boundary layer turbulence; and finally 5) the transmission of vibrational and acoustical energy in complex structures.

By describing a series of detailed analyses we do not want to leave the reader with the impression that only ideal structures are being considered. On the contrary, it is because there is much evidence, both theoretical and experimental, that the average behavior of ideal and non-ideal structural configurations are very similar, that an ideal structure can be substituted for the real system that we are interested in analyzing. Thus, for example, modal densities have been computed for flat plates assuming supported boundary conditions. The assumption is, that if the modal density depends on the area alone, another panel with different boundary conditions but possessing the same area will have sensibly the same modal density. There is now considerable experimental evidence that such is the case.

In this connection there are several other parameters which one would like to make similar hypotheses about, but there is at present little direct confirmation that this can be done. For example, in the preceding chapter we found there were modes which radiated like strip radiators along the edges of the panel with a fairly high radiation efficiency, while other modes radiate only at the corners of the panel with fairly low radiation efficiency. In real structures with non-idealized boundaries, it may be that a particular mode shows combination of such radiation behavior rather than simply one form or the other. Experiments suggest that the average radiation behavior nevertheless may still be computed by the idealized model. There is, however, no theoretical evidence that will support such a conclusion. Its experimental justification, however, makes it very appealing as a way of calculating the radiation from non-ideal panels for engineering purposes.

VII.2 Acoustic Coupling to Flat Panels and Beams

A fundamental study of structure sound interaction which parallels closely the material presented in Chapter V has been presented in a paper by Smith[1,*]. The response of a simple resonator representing a single mode of motion to both pure tones and band of noise are presented. Reciprocity arguments similar to those in Chapter V are used to establish the relation between the energy of response, radiation resistance, and directivity factors. It is not brought out in Smith's article, but it was demonstrated in Chapter II that when many modes of a structure are simultaneously excited to a similar degree, then the noise and pure tone response of the structure are approximately equal on the average. An item discussed by Smith in his paper but not presented in this report is the reradiation of sound, formulated as a resonant differential scattering cross-section. In an expression for the cross-section, the directivity factor enters twice, once in expressing the energy absorbed by the incident sound waves and again in expressing the fractional amount of energy re-radiated in the direction of interest.

Another approach to the problem by Lyon and Maidanik[2] regards the interaction between a sound field and a structure as a collection of two mode interaction problems. In this interaction, one mode is a structural mode and the other is a mode of the room within which the sound field is confined. An

*In this chapter only, references occur in a numbered bibliography at the end of the chapter.

analysis of the two mode interaction problem in this reference reveals that the energy flow from one mode to another has certain analogies with a simple heat flow problem, primarily in that the magnitude of heat flow is proportional to the difference in modal energies, and that the direction of flow is from the mode of higher energy to one of lower energy. Normally the modal densities will be such that a single structural mode at any one time is in contact with a large number of acoustic modes. If these acoustic modes have similar energies, which is the modal equivalent of a diffuse sound field, then one can readily collect these interactions together to produce response formulas such as given in Chapter V. The details of the transformation between a two mode interaction and a multi mode interaction are developed in the referenced paper. Particular emphasis is placed on the interaction between a sound field and a structure, although the general results are applicable to any two systems that can be defined as a collection of normal modes of motion.

In this reference the radiation resistance is expressed in terms of an inner product of the correlation functions of the sound field and the structural motion. The reader may wish to test the equivalence of this and the development of Chapter V as an exercise. Experiments are described in which the response of a simply supported beam to a reverberant sound field is measured. The modal response in the beam is compared with theoretical calculations. The calculations involve theoretical computations of the radiation resistance of a simply supported beam and measured values of the internal loss factor. The radiation from a simply supported beam is of interest in the panel response case as well since in Chapter VI we saw that many panel modes radiate as though they were simply strip radiators along one panel edge.

Above the critical frequency the beam wavelength exceeds the acoustic wavelength, and an expression for the radiation resistance similar to that developed for edge modes in Chapter VI is derived. Below the critical frequency, the beam wavelength is less than the acoustic wavelength and radiation from the ends due to uncancelled volume velocity results giving a radiation efficiency similar to that of the corner modes discussed in Chapter VI.

In another paper Maidanik[3] has computed and measured experimentally the response of ribbed panels to reverberant sound fields. Maidanik uses the general expressions for the response of a multimodal structure exposed to reverberant sound fields developed in reference 2 and addresses himself in this paper primarily to the estimation of the radiation resistance of a simply supported rectangular panel. We presented the analysis in Chapter VI for a rectangular supported panel when the dimensions are large compared to an acoustic wavelength. This assumption simplifies the analysis somewhat in that the interactions between the uncancelled volume velocity elements at the edges or corners can be treated independently. When the panel size is smaller than an acoustic wavelength these simple radiators interact and in this paper Maidanik has developed rather carefully the effect of this interaction.

Maidanik comes to the conclusion, as we did in Chapter VI, that below the critical frequency the sound radiation may be directly associated with the edges of the panel. The radiation efficiency then becomes proportional to the perimeter of the supported panel or in the case of a large panel with many ribs to the total edge perimeter produced by the panel edges and the ribs. To support this conclusion Maidanik cites the results of some previous experiments and carries out some new experiments which are reported in the paper. In particular he applies the results of his theoretical calculations to the radiation efficiency of a large aluminum panel with semi-clamped edges which was measured several years before. His own experiments were carried out on a large panel with free edges, baffled, and sealed with tape to prevent acoustic leaks from one side of the panel to the other. In addition to this several steel beams were cemented to the panel to produce additional structural discontinuities. In both cases Maidanik finds experimentally that the radiation efficiency of the panel may be accounted for through the simply supported edge condition assumed in the analysis. These results suggest a certain insensitivity of the radiation efficiency to the details of the edge condition, although as we shall see, analyses by Lyon and Smith have indicated some sensitivity of the radiation efficiency to edge conditions. Whether one can significantly affect the radiation efficiency by careful design of edge discontinuities is still an open question.

The study of finite panels has suggested that the major source of radiation below the critical frequencies arises from the interaction of the bending wave with an edge discontinuity. In particular, those modes of motion which result in a trace wavelength along the edge greater than the acoustic wavelength are responsible for most of the radiation. In order to look at the effects of varying edge conditions on this source of radiation without the complexity or difficulty of finding normal modes for arbitrary boundary conditions, a study was initiated of the acoustic power radiated when a beam is placed in contact with a plate supporting a reverberant vibrational field of acoustically slow bending waves.[4]

In this analysis the beam is defined by its torsional and bending rigidities and its mass per unit length. A simply supported line is characterized by infinite mass and bending rigidity and vanishing torsional rigidity. A clamped line is characterized by infinite bending and torsional rigidities. Note that neither condition is equivalent to the edge support of a plate since in that situation the plate does not continue beyond the boundary and there is no moment impedance presented to the plate at the edge. The results of this analysis show that the radiation resistance per unit length of perimeter for a panel is the same for a simply supported and a clamped edge but that the radiation resistance is just twice that of an edge-supported panel analyzed by Maidanik. It has also been suggested by some that the radiation from a panel with supporting ribs should undergo a marked transition at the critical frequency of the supporting rib. An analysis of this effect also presented in the paper indicates that as long as the mass of the rib is sufficiently large to restrict the motion of the panel the critical frequency of the supporting rib is not an important parameter in the problem. Experimental evidence does suggest that when the bending wavelength on the panel becomes large enough so that the ribs begin to move with the panel then the radiation efficiency of the structure is significantly decreased.

Another extension to edge radiation analysis has been made by Smith[5] who calculated the radiation due to a simple bending wave normally incident on a supported line with arbitrary moment impedance. Smith's results agree with those of Lyon and Maidanik in that the radiation from a clamped edge is found to be just twice that due to a simply supported edge with no moment communication. He also found, however, that the radiation from such a line can be made to assume many intermediate values as the impedance parameter is varied and in particular can be made to vanish for a particular value of mass reactance. This analysis is arrived at by considering the volume velocity cancellation and in fact the result of vanishing radiation efficiency results from a net uncancelled volume velocity of zero at the edge. This condition holds only at one frequency, however, and only for normal angles of incidence. Whether one can minimize the radiation of a real panel by a suitable selection of edge impedance remains to be seen.

VII.3 Response of Cylindrical Structures to Sound Fields

Many structures of engineering interest such as aircraft, spacecraft, and submarines are constructed of stiffened cylinders. Since the acoustic response and radiation behavior of these structures is of considerable engineering interest one of the important extensions of the study of flat panels described in the previous section has been to include the effects of curvature on the radiation and response behavior. Heckl[6] studied the modal density of cylindrical shells and the average impedance which they present to a point force perpendicular to the surface. Some of his results that are pertinent to the acoustic response problem are: a) in modal density and average mechanical behavior the cylindrical shell tends to act like a flat plate above the so-called ring frequency, i.e. the frequency at which the circumference of the shell is equal to a longitudinal wavelength in the shell material; b) below this frequency the modal density is an increasing function of frequency with an average input impedance greater than that of the flat plate of equivalent area. Heckl confirms these theoretical results with a series of experiments in which he measures individual resonant frequencies and also counts the accumulated number of modes to derive the modal density. For the cylinders he used it was only possible to measure modal density below the ring frequency.

In a paper which parallels the approach in reference 3 Manning and Maidanik[7] have computed the theoretical radiation efficiency of a finite simply supported cylinder with baffled ends and compared these calculations with the measured radiation from a steel cylinder with circumferential stiffening frames. Experimentally the radiation efficiency of the finite cylinder shows some similarities to that of the finite supported plate. Above the critical frequency the radiation efficiency is nearly unity and it drops off below the critical frequency. At the ring frequency, however, there is an abrupt peak in the radiation efficiency and an octave or so below the ring frequency there is a sudden dropoff in radiated acoustic power.

In their analysis, the authors are able to justify this behavior by a consideration of the following effects. First, because a cylinder is like a flat plate which is closed upon itself there is no radiation corresponding to the "corner modes" since now the uncancelled volume velocity patches are adjacent to each other. Secondly, since there are no axial discontinuities on the cylinder the edge radiation corresponding to axial trace wavelengths greater than the acoustic wavelengths do not occur. Modes which have circumferential trace wavelengths greater than the acoustic wavelengths are moving in a nearly axial direction and consequently are significantly affected by the stiffening effects of curvature. The net tendency is for an upward shift in their resonance frequency which depletes the lower frequency region of radiating modes. This is the reason for the abrupt dropoff in radiation at low frequencies. These modes which are shifted upwards tend to collect at the ring frequency with a helical wave speed in excess of the speed of sound. Thus although they occur at a frequency below the panel critical frequency, they are acoustically fast and have a high radiation efficiency. This is the reason for the peak in radiation efficiency at the ring frequency.

The relative number of acoustically fast and circumferential edge radiating modes is determined by the ratio of the ring frequency to the acoustic critical frequency. When these are within an octave or so of each other the acoustically fast modes tend to dominate the radiation behavior. When this is the case the entire surface of the structure radiates sound and the edge discontinuities are not of great significance. Any axial stiffening, however, due to stringers will affect the distribution of acoustically fast modes and the radiation behavior will be affected. When the ring frequency is far below the critical frequency, as it is for the structures mentioned at the beginning of this section, then circumferential edge radiation is a significant contributor to the total radiation efficiency, and the number of ring supporting frames will have a bearing on the radiation behavior of the structure.

The decrease in radiation efficiency at very low frequencies is experimentally not as great as anticipated from the theoretical analysis. This appears to be due to forced radiation by nonresonant modes which are excited below their resonance frequencies. Manning and Maidanik have made an estimate of the forced radiation from a cylinder in the reference but the calculation is appropriate to a flat plate, where the forced radiation occurs due to modes which are excited above their resonant frequencies and act therefore in a mass controlled fashion. The proper radiation behavior for forced motion has not been carried out for the cylinder to date.

Two estimates have recently been made of the response of a space vehicle to an acoustic noise field. Both approaches use the energy technique developed in the preceding chapters and to some extent both rely on the structural configuration of the spacecraft for evaluation of radiation resistance and internal damping parameters. Neither one, however, takes the effects of curvature on the structural mechanics into account in such detail as does the discussion in reference 7.

In the first of these analyses, Franken and Lyon[8] describe the vehicle as a combination of long, narrow, flat panels. The effects of curvature are not included in the analysis. The long panels are generated by infrequent ring stiffeners and a large number of axial stringers which give stiffening to the skin. The radiation response is produced by edge radiating modes in the axial direction. The geometry of the panel produces a "clumping" of modes at panel cross-resonances in the smaller dimension. There is some evidence from the experimental values which are cited which indicate that peaks in the response are produced at these cross-resonance frequencies.

Estimates of absorbed power from the sound field are made for three different situations: a purely axial travelling wave field, a purely diffuse wave field, and a weighted distribution which emphasizes energy at angles near grazing incidence for the panels. The analytic results do not indicate any significant difference in the last two distributions of incident energy, but for a purely axial wave field, larger peaks in the energy spectrum are produced at the cross-resonance frequencies. The results of the calculation are in general agreement with the field data, but they cannot of course predict the general rise in energy levels near the ring frequency and they suffer from a certain arbitrariness because of the necessity to assume an internal damping.

An analysis of a similar structure, the second study referred to above, by Dyer[9] attempts to take into consideration more fully some of the effects of curvature. Dyer uses the flat plate edge mode radiation efficiency derived in Chapter VI and assigns to the edges of the panel a certain average absorption coefficient. By doing this, the ratio of radiation resistance to total damping becomes independent of the perimeter. The modal density of the structure, however, is that of a cylinder as derived in reference 6 by Heckl, and it is this value that he uses for the structural modal density. The other effect of curvature is his assumption that the sound field on different segments of the structure is uncorrelated due to shielding effects of the structure. The incident acoustic energy is assumed to arrive over a small cone of angles near grazing. He does not include the possibility of modal clumping at the cross-resonances of the panels as was done by Franken and Lyon. He compares his theoretical results with the same field data used in reference 8, with a general agreement between experimental and theoretical results.

The relative success of these rather different calculations suggests a certain insensitivity of the response to the details of the behavior of the radiation efficiency. There is, however, in the field data a notable tendency for the energy to have a maximum near the ring frequency at 500 cycles per second. It therefore appears necessary to include the effects of curvature on modal dynamics and the regrouping of edge radiating modes as Manning and Maidanik have done if a better estimate of the radiation of cylindrical structures is to be achieved.

VII.4 Fluid Loading Effects on Panel and Response and Radiation

The radiation of sound by vibrating structural panels in contact with a heavy fluid medium is of concern to the designers of ships and submarines. Basically, these structures are similar to aircraft and spacecraft vehicles in that they consist of flat panels stiffened by reinforcing ribs. The primary difference occurs because of the very large influence that the fluid reaction pressures produce on the mechanical and radiation behavior of the structures. Recent studies by Maidanik and Kerwin[10] on the response and radiation of orthotropic plates including fluid loading effects have shed considerable light on the behavior of these panels.

The structural model that Maidanik and Kerwin have chosen is an orthotropic plate with differing bending rigidities in the two major directions. The effects of curvature as such are not specifically included in the discussion. The fluid loading for a specific frequency and wavelength of motion on the panel, not necessarily the wavelength corresponding to free-travelling waves, is computed by solving the fluid acoustic wave equation and calculating the reaction fluid pressure on the panel. At any frequency, wavelengths on the panel greater than the acoustic wavelength have a real, resistive load placed on them by the fluid. Wavelengths less than the acoustic wavelength are loaded by a mass-like term resulting from near-field, nonradiated acoustic pressures.

With heavy fluid loading these reaction pressures have the effect of heavily damping wave motions greater than the acoustic wavelength and of adding considerable mass to the effective plate mass at wavelengths smaller than the acoustic wavelength. This mass reaction at a particular frequency has the effect of slowing down the free wave on the panel, often to a sizable fraction less than would be true for the same panel in vacuum. Because of these slower free waves the resonance frequencies of finite panels are reduced. At any particular frequency the wavelength of motion is reduced. The radiated power is therefore affected in two ways. First, the number of radiating modes in any frequency interval is increased because mass load increases the modal density of the plate. Secondly, because the bending wavelength is reduced, the radiation efficiency of edge modes will be reduced since the effective width of the strip radiators along the edge is reduced.

At any frequency, the effective fluid loading on the power absorbed by a point source, which is of concern for impact loading and for boundary layer noise studies, is to reduce the power absorbed by the panel. This is because the input point susceptance is reduced. This arises from two causes: one is that the input point resistance is increased by the fluid loading and the second is that an additional mass reactance term becomes effective. There are many aspects of the fluid loading problem remaining to be investigated, and they are at present receiving considerable attention. The problem is inherently more complicated than that of light fluid loading, but the energy method has enabled one to compute many of the important effects already, and there is reason for optimism that one can do equally well in the future.

VII.5 Response and Radiation of Panels Excited by Boundary Layer Turbulence

The response of panels which are excited by pressure fluctuations in boundary layer turbulence and the resulting sound radiation from these panels, both inward to the structure interior and outward to the medium, are a present day concern. The ship designer is concerned because of the sound radiation external to the hull of his vessel and internal to the sonar dome onto pressure transducers associated with sonar receivers. High speed aircraft and manned vehicles have high internal sound levels due to the very high pressure fluctuations which result in high speed flight. Despite the very wide differences in the two sets of problems it is possible to treat them with similar basic analyses, and it is the development of these analyses which we wish to trace in this section.

It has become common practice to describe the pressure fluctuations in boundary layer turbulence by a convecting and decaying correlation field. The response of a simple string to such a correlation field was first studied by Lyon[11]. In that study several features were apparent in the response which have turned out to be significant in the behavior of panels excited by similar convected pressure patterns. When the convection speed is quite low compared to the speed of the structural wave the response is that due to a series of random independent impulses on the structure, the so-called "rain on the roof" excitation. At higher speeds when the convection speed can equal the structural wave speed, a very large response will result if the distance over which the excitation is convected coherently is a sizable fraction of a structural wavelength. This condition does not occur for ship structures, but can be quite important for aerospace structures.

The one-dimensional treatment has since been further modified and expanded by Maidanik[12] and by Maidanik and Lyon[13]. The study of reference 12 was undertaken because of the reluctance on the part of some people to accept delta function spatial correlation as appropriate to boundary layer turbulence as was proposed in reference 11 and used subsequently for panel response by Dyer[14]. Maidanik computed the response of a string to several forms of spatial correlation which could be made to go to a delta function by a limiting process of a shape parameter. As might be expected, in every case he found that as long as the spatial correlation was small compared to the structural wavelength the delta function approximation gave the proper result. He did not consider specifically the problem of whether or not the actual spatial correlations in boundary layer turbulence could be approximated by any function which could be made to approach the delta function with the variation of some parameter. It now appears that it is impossible to make such a correspondence since the pressure correlation in boundary layer turbulence appears to have zero net area.[15] This causes no fundamental difficulty as we shall see in our discussion of reference 15. Maidanik does extend the earlier results by not assuming that the correlation pattern decays in a distance small compared with the length of the string and consequently has expressions which include explicitly the effects of the turbulent pressure fluctuations passing over the edge of the structure. This tends to be an academic point for most structures since the distance between edges is large compared to the distance that the correlated patch travels before decay. The discussion in reference 13 is primarily aimed at refining the estimates of modal response in reference 13 and forming a new comparison with the experimental results which were reported in the earlier paper.

The application of this pressure correlation pattern and structural model to flat elastic panels was carried out by Dyer[14]. Dyer assumes that the distance over which the coherent patch is convected is small compared to the length of the panel over which the turbulence flows. In the analysis it turns out that the results are very similar to those developed for the string, if one interprets the structural wave speed as the trace wave speed of the particular mode under investigation in the direction of convection. Dyer studied the response of the panel to turbulent excitation when the trace wave speed was greater than equal to and less than the speed of turbulent convection. Since then these three classes of modes have been termed hydrodynamically fast, hydrodynamically coincident, and hydrodynamically slow. Dyer computes the mean-square response of a mode and plots its dependence on the damping loss factor. He finds that for high frequencies or slowly decaying turbulence that the mean-square response can become dominated by its forced components and be independent of the loss factor. One should be cautioned against interpreting this result as an insensitivity of the response to damping since the panel mode may still be highly resonant and, in the frequency range of its resonance, have a response which is very markedly dependent on the damping.

In a more recent report, Ffowcs-Williams and Lyon have considered this problem again, taking into account a better understanding of the pressure correlation pattern and of the sound radiation behavior of multi-modal flexible panels. The considerations of correlation effects indicate that the effective correlation area of the turbulence should be considered to be a function of the ratio of displacement thickness of the turbulent boundary layer to the free bending wavelength of the structure at the frequency of interest. It turns out that previous estimates of this relationship have been reasonably correct, but over a range of 3 decades in frequency approximately a 10 db variation can result from this correlation effect.

The radiation of sound from the panel is handled in two parts. The radiation from forced waves or non-resonant panel motion at wavelengths greater than the acoustic wavelength is computed on the basis of an infinitely extended plate. The frequency spectrum and overall power level for this portion of the radiation is determined. The radiation from resonant modes below the critical frequency is computed based on the radiation efficiencies developed in Chapter VI and in reference 3, all calculations being carried out for the case of hydrodynamically fast modes, which means high frequencies and/or flow convection speeds.

The mechanical power absorbed by a panel from turbulence convected at a speed greater than the panel bending wave speed has been studied by Ribner[16] and Lyon[17]. Ribner computed the panel response due to convected and non-decaying turbulence and found, as one would expect, that free wave excitation of the panel resulted only for waves whose bending speed was greater than the speed of convection, which places a high frequency limit on the excitation spectrum. By retaining the decaying formalism, Lyon[18] was able to place a criterion on the distance over which the eddy must remain coherent in order for the turbulence to behave in an essentially non-decaying fashion. It turns out that this is quite a long distance and that in practice the lower frequency part of the spectrum will accordingly be independent of the decay parameter and determined primarily by the hydrodynamically coincident wave field.

VII.6 Transmission of Vibrational and Acoustic Energy in Multi Element Structures

In this section we shall briefly summarize two related areas of vibration and sound transmission technology. The first is the exchange of energy among purely mechanical systems. When only two systems are involved, the energy sharing process is quite similar to coupling between a sound field and a structure. When more than two elements are involved, particularly when there is a transmission of energy through one element, then the formalism has to be recast and the previous discussion extended. This topic is also considered here. Finally, an interesting study of the relationship between impact noise and acoustic transmission loss is included as an example of the value of reciprocity "thought experiments" in the analysis of complex acoustic and mechanical interaction problems.

Throughout the preceding discussion we have seen the central role that the radiation resistance of a structure plays in its interaction with an acoustic field, both in determining the amount of power radiated by the vibrating structure and in determining the amount of power it will accept from a diffuse reverberant sound field. It has been demonstrated by Lyon and Eichler[18] that a similar relationship holds between two mechanical structures which are coupled. Starting with the general considerations of reference 2, they developed the expressions for the interaction between beams and plates and between attached plates, resulting in expressions for the energy ratios in the two attached structures when one of them is driven by random noise source. In this analysis the coupling loss factor, which can be related to the input impedance function of the two structures at their region of attachment, plays the part of the radiation resistance, while the internal damping of the structures is represented by internal loss factors. Experiments are described and carried out in which the energy ratio of a plate and beam and of two plates welded together are measured. Good agreement with the theory is found for broad bands of noise when many modes of vibration participate in the energy-sharing process.

When the bandwidth of excitation is more restricted, fewer modes contribute to the response and a larger amount of variability from the theoretical average is determined experimentally. Using a statistical model for the excitation levels of the structural modes and the location of resonance frequencies of the structure, the variability in response levels is computed and presented in the form of curves of the confidence coefficient for certain interval estimates of the response ratio.

The earliest study of transmission through a multi element system was that presented for the noise reduction of a flexible panel due to Lyon[19]. In this paper the noise reduction afforded by a single flexible panel on a rectangular box is computed in three different frequency ranges. At very low frequencies the panel and the enclosed volume of the box behave in a stiffness controlled fashion, and the noise reduction, i.e., the logarithm of the ratio of external to interior pressure is frequency independence. In a slightly higher frequency range the interior volume is still stiffness controlled but the panel modes now come into play. Those which have a net volume velocity displacement produce pressures within the box due to compression. In this frequency range the noise reduction fluctuates greatly and may in fact become negative at some frequencies, since pressure amplification can result.

Finally, in the high frequency range where the panel and the interior box volume are multi modal in behavior one can compute the noise reduction on the basis of "classical" forced wave theory, as well as the interior sound pressures due to resonant panel vibration and edge and surface radiation. In this regime one cannot say generally whether forced or resonant noise reduction will dominate, but for small panels it may frequently occur that resonance vibration can cause smaller amounts of noise reduction than would be anticipated under the classical theory.

The analysis in reference 19 does not include properly the back reaction of the interior volume on the panel. That is, the build-up and level in the interior volume is not assumed to affect the rate at which energy is accepted by the panel from the exterior sound fields. A more complete analysis of the energy sharing in multi stage systems has been carried out by Eichler[20]. Eichler develops a general formalism which parallels the development in reference 2 for multi stage systems and applies his analysis to the noise transmission through a rectangular box with flexible walls. Due to his choice of parameters and available experimental conditions he is not able to show a significant discrepancy between his more complete analysis and the approximate analysis of reference 19. His experimental results, however, do represent the most complete study of the noise reduction of small metal boxes which has been reported to date in the literature.

When a panel vibrates at frequencies above its critical frequency the input impedance from a point source can be related to the acoustic transmission because both rely on the excitation of free waves, in one case by the point impact and in the other by a reverberant sound field. Using a reciprocity argument Heckl and Rathe[21] have been able to compute a simple relationship that should exist between the transmission loss of a structural panel and the impact noise isolation of that same panel. Independent measurements of impact noise level and transmission loss can therefore be used to detect anamolous behavior, i.e., departure from the ideal relationship. This will occur either because the free wave relationship is disturbed by a resilient material placed under the impacting point or because flanking transmission occurs in the acoustic transmission loss measurements. When this happens, the wall in question is not the only source of energy in the second space.

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APPENDIX I: AVERAGES OVER DIRECTION OF WAVE INCIDENCE

In assessing the average coupling between sound waves and vibration of plane structures, one desires to form averages over the direction of incidence of the wave. Typically, the response to a pure-tone plane wave, with wavenumber vector $\underline{k} = k\underline{\Omega}$ (where $\underline{\Omega}$ is a unit vector), is a function $F(\underline{k}_s)$ where $\underline{k}_s$ is the vectorial projection of $\underline{k}$ onto the plane of the structure (the "trace wavenumber"). The unit vector $\underline{\Omega}$ specifies the direction of wave propagation. One wants the average of F for all directions corresponding to incidence from above the plane (2π sterradians of solid angle) with each direction given equal weight. The average over direction $\underline{\Omega}$ must be transformed into an average over the trace wavenumber $\underline{k}_s$ on the plane.

The average value of F is formally equivalent to the ratio of integrals

$$\langle F\rangle_{\underline{\Omega}} = \int F \, d\underline{\Omega} \Big/ \int d\underline{\Omega} \quad , \qquad \text{(AI.1)}$$

where $d\underline{\Omega}$ denotes the differential of solid angle, equal by definition to the differential of area on the unit sphere. When the $\underline{k}$ vector is expressed in polar coordinates (k,θ,ϕ) with the $\overline{z}$ axis (the normal to the plane) as polar axis (Fig. AI.1), the differential of solid angle is

$$d\underline{\Omega} = \sin\theta \, d\theta \, d\phi \ .$$

Let the trace wavenumber $\underline{k}_s$ be expressed in plane polar coordinates (k_s,ϕ) where

$$k_s = k \sin\theta \ . \qquad \text{(AI.2)}$$

Then, one can readily show that the transformation of coordinates from $\underline{\Omega}$ to $\underline{k}_s$ is described by the differential relation

$$k^2 \cos\theta \, d\underline{\Omega} = k_s \, dk_s \, d\phi \equiv d\underline{k}_s \ ,$$

where $d\underline{k}_s$ is the differential of area in the $\underline{k}_s$ plane. Equation AI.1 can be rewritten

$$\langle F\rangle_{\underline{\Omega}} = \int [F/\cos\theta] \, d\underline{k}_s \Big/ k^2 \int d\underline{\Omega} \quad . \qquad \text{(AI.3)}$$

For wave directions limited to incidence on one side of the xy plane, one has

$$\int d\underline{\Omega} = 2\pi \text{ ster-radians} .$$

Moreover, to each direction Ω there corresponds one $\underline{k}_s$; its magnitude does not exceed k (Eq. AI.2). The whole range of $\underline{\Omega}$ for incidence from one side corresponds to the whole range of $\underline{k}_s$ lying within the circle of radius k in the $\underline{k}_s$ plane (Fig. AI.1); for these ranges, one has

$$k^2 \int d\underline{\Omega} = 2\pi k^2 = 2\int d\underline{k}_s \quad .$$

Equation AI.3 can now be rewritten as an average in the $\underline{k}_s$ plane:

$$\langle F(\underline{k}_s)\rangle_{\underline{\Omega}} \text{ (one side)} = \frac{1}{2} \langle F/\cos\theta\rangle_{\underline{k}_s} (0<k_s<k) \qquad \text{(AI.4)}$$

where $\cos\theta$ is derivable from Eq. AI.2. The parenthetical subscripts indicate the range of the variables.

In some situations, $F(\underline{k}_s)$ depends only on one cartesian component of $\underline{k}$. This is the case when the structure is a beam. Suppose F is a function only of k_x, the projection of $\underline{k}$ onto the x axis. Then, Eq. AI.4 can be reduced to an average in k_x. However, the derivation is more convenient if the $\underline{k}$ vector is expressed in new polar coordinates (k,α,β) wherein α is the polar angle measured from the x axis (Fig. AI.2). Then the differential of solid angle is

$$d\underline{\Omega} = \sin\alpha \; d\alpha \; d\beta \; .$$

Since the trace wavenumber k_x is given by

$$k_x = k \cos\alpha \; .$$

the transformation of coordinates is described by

$$k d\underline{\Omega} = - dk_x \; d\beta \; .$$

Wave directions for incidence from one side of the xy plane correspond to the ranges of variables:

$$0 \leqslant \alpha \leqslant \pi \; ; \; +k \geqslant k_x \geqslant -k \; ; \; 0 \leqslant \beta \leqslant \pi \; .$$

Since F is independent of β, that part of the integration in Eq. AI.1 can be carried out, leaving the desired expression:

$$\langle F(k_x)\rangle_{\underline{\Omega}} \text{ (one side)} = \pi \int_{-k}^{k} F \; dk_x/2\pi k = \langle F\rangle_{k_x} (-k<k_x<k) \cdot \qquad \text{(AI.5)}$$

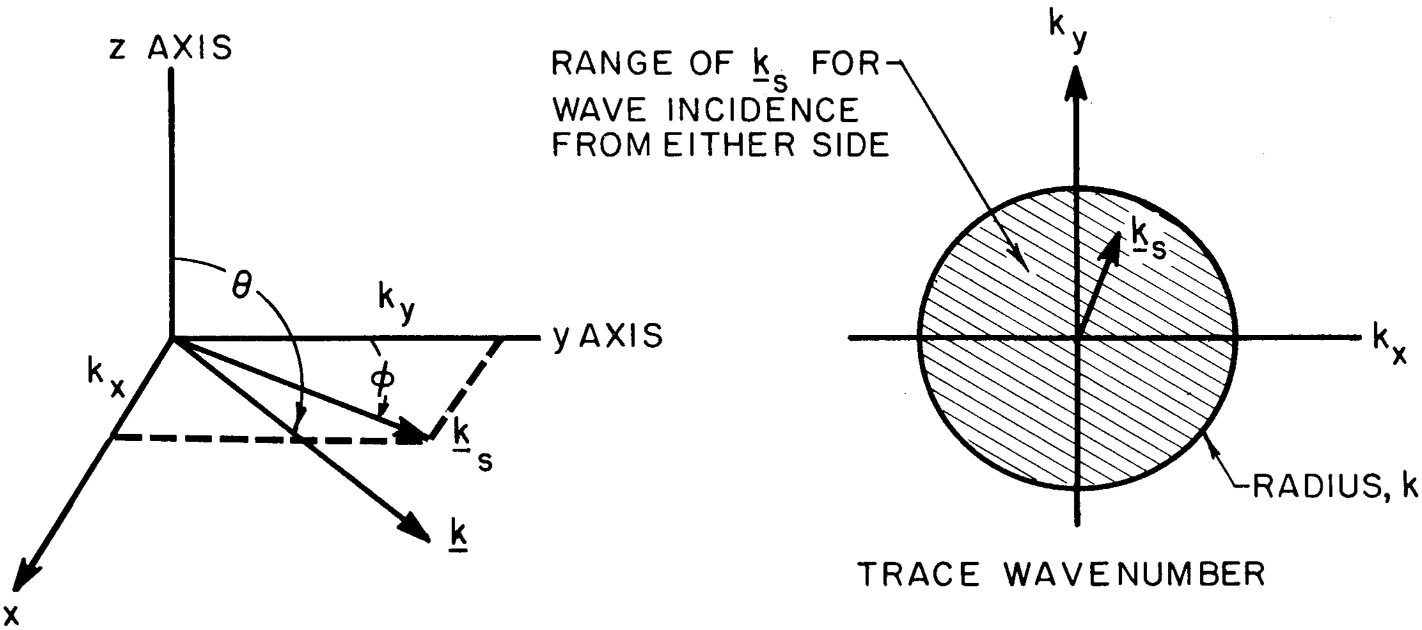

Figure AI.1.- Wavenumber plots; trace wavenumber for xy-plane.

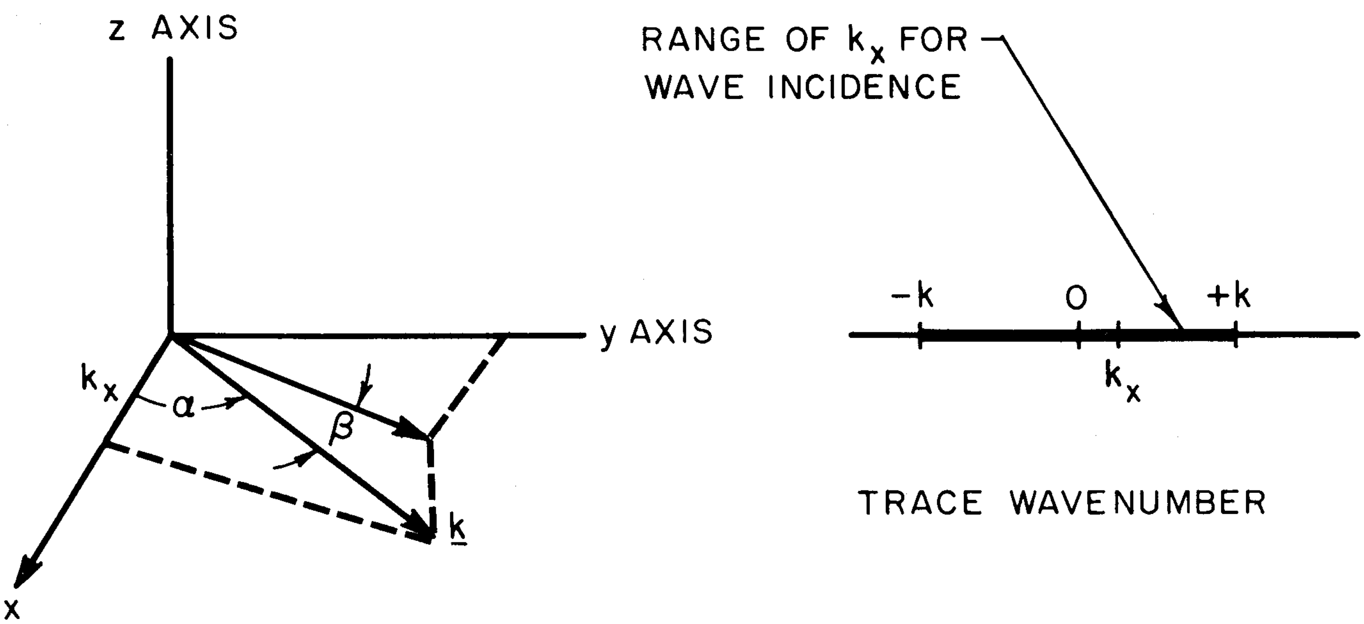

Figure AI.2.- Wavenumber plots; trace wavenumber for x-axis.

APPENDIX II: EVALUATION AND APPROXIMATION OF A COUPLING INTEGRAL

The analysis for coupling between sound pressures and vibration of plane structures (panels or beams in a plane baffle) requires values for integrals in the form:

$$I = \ell^{-1} \int_0^{\ell} e^{-ik_x x} \sin k_m x \, dx \quad , \tag{AII.1}$$

where $k_m \ell = m\pi$ with m an integer. More specifically, one desires to evaluate $|I|^2$ and weighted definite integrals of it in the form

$$\int_a^b K(k_x) \; |I|^2 \, dk_x \; , \tag{AII.2}$$

where K is some smooth, well-behaved function. Here we evaluate I and develop some approximations for the weighted integral.

When the exponential form for $\sin k_m x$ is substituted in Eq. AII.1, the integrand contains two exponential terms, each of which is integrated readily. If each integral is treated separately, the value of I is found as a sum of two parts

$$I = I_1 + I_2 \; ,$$

$$I_1 = i\exp[-i\tfrac{1}{2}(k_x+k_m)\ell\,]\frac{\sin \frac{1}{2}(k_x+k_m)\ell}{(k_x+k_m)\ell} \; ,$$

$$I_2(k_x,k_m) = -I_1(k_x,-k_m) \; . \tag{AII.3}$$

The identity

$$1 - e^{-ia} = 2ie^{-i\frac{1}{2}a} \sin\tfrac{1}{2}a \quad .$$

When the restriction, $k_m \ell = m\pi$ with m an integer, is introduced into Eq. AII.3, a closed form can be found for the sum of the two terms. The form is different for even and odd values of m. The magnitude of I is

$$|I| = \left| \begin{matrix} \sin \\ \\ \cos \end{matrix} \; \frac{1}{2} k_x \ell \Big/ \frac{1}{2} k_m \ell \; (1 - k_x^2/k_m^2) \right| , \qquad \text{(AII.4)}$$

where the sine function is to be used if m is even, and the cosine if m is odd. The ratio is indeterminate at $k_x^2 = k_m^2$, at which points the correct value

$$|I|_{k_x^2 = k_m^2} = \frac{1}{2} \qquad \text{(AII.5)}$$

is found by l'Hopital's rule.

As a typical case, take a large value of m. The squared magnitude, $|I|^2$, is an even function of k_x which oscillates between zero and an envelope determined by the denominator of Eq. AII.4. (See Fig. AII.1.) The envelope is flat near the origin ($k_x^2 << k_m^2$), rises to a singularity at $k_x^2 = k_m^2$, and falls rapidly for larger values of k_x^2. At the singularity of the envelope, $|I|^2$ rises to a finite maximum value (Eq. AII.5) which is much larger than other local maxima. For example, the next largest extrema are the minor peaks neighboring the major peak, which occur at

$$k_x \ell \approx k_m \ell \pm 3\pi;$$

it follows readily from Eqs. II.4 and II.5 that the value of $|I|^2$ is there about 1/22 of the value at the major peak.

<u>Analytical Approximations</u>

In a weighted integral of $|I|^2$, such as Eq. AII.2, rather simple approximations can be used if the weighting function $K(k_x)$ is slowly varying in comparison with $|I|^2$. Different approximations are appropriate to different ranges of k_x. The following approximations are good for large values of m.

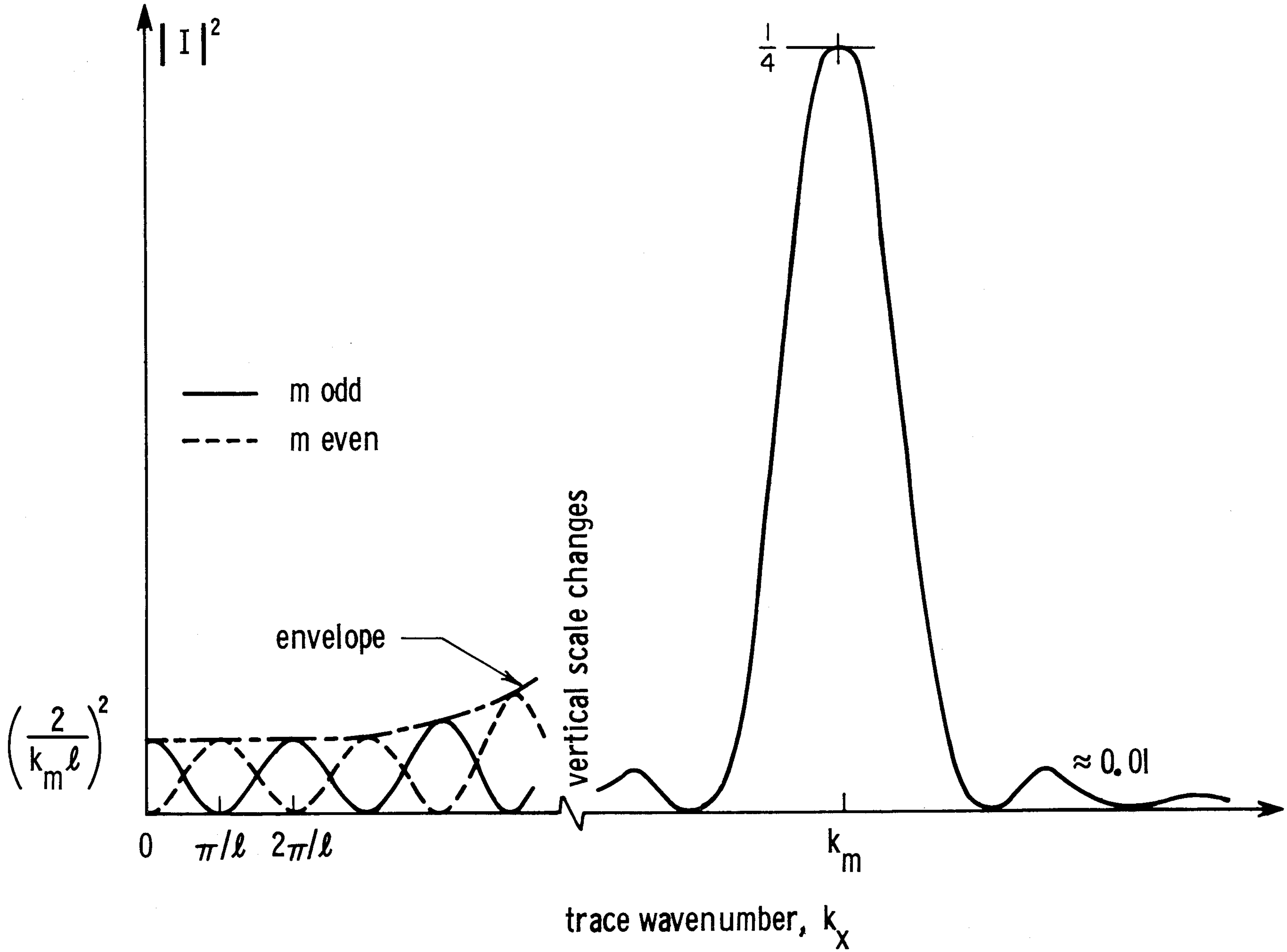

Figure AII.1.- Sketch of $|I|^2$ for large mode numbers m.

1) $k_x^2\ell^2 < 1$: In a narrow range near the origin, $|I|^2$ is approximately constant. The analytical approximation is different for even and odd values of m:

$$\int K|I|^2 \, dk_x \approx \int K \, dk_x \times \begin{Bmatrix} 0, \; m \text{ even} \\ 4/k_m^2\ell^2, \; m \text{ odd} \end{Bmatrix} . \qquad \text{(AII.6)}$$

2) broad range of $k_x^2 << k_m^2$: In an integral over a range of k_x that includes several of the minor peaks, one may approximate $|I|^2$ by one-half of the value of its envelope, since the average value of both $\sin^2$ and $\cos^2$ over each single loop is 1/2. No distinction between odd and even values of m is necessary. If, further, $k_x^2 << k_m^2$, the envelope can be approximated by a constant; this further approximation is fairly good if $k_x < \frac{1}{2} k_m$:

$$\int K|I|^2 dk_x \approx (2/k_m^2\ell^2) \int K \, dk_x, \text{ all m.} \qquad \text{(AII.7)}$$

3) range of k_x including k_m: When the range of k_x includes one of the major peaks of $|I|^2$, either $\pm k_m$, the value of the integral is usually dominated by contributions from the neighborhood of that peak. This situation is demonstrated by the uniformly weighted integratal,

$$\int |I|^2 \, dk_x .$$

When the limits of integration include only the major peak, i.e.,

$$k_m\ell - 2\pi \leqslant k_x\ell \leqslant k_m\ell + 2\pi .$$

The value of the integral is 90 percent of the integral over the semi-infinite range,

$$\int_0^\infty |I|^2 \, dk_x = \pi/2\ell . \qquad \text{(AII.8)}$$

These characteristics of $|I|^2$ suggests that it can usually be approximated by a pair of δ-functions of equal strength,

$$|I|^2 \approx (\pi/2\ell)[\delta(k_x-k_m) + \delta(k_x+k_m)] \ , \qquad \text{(AII.9)}$$

if the range of integration includes either $\pm k_m$ and the weighting function $K(k_x)$ does not have large peaks at other points. The corresponding integral approximation is

$$\int_{-k_m<k_x<k_m}^{k_x>k_m} K|I|^2 \, dk_x \approx (\pi/2\ell) \ K(k_m). \qquad \text{(AII.10)}$$

A similar approximation applies to integrals including $k_x = -k_m$.

4) <u>broad range of $k_x^2>>k_m^2$:</u> In an integral over a broad range of large values of k_x, not including either major peak of $|I|^2$, an approximation similar to the second approximation above is appropriate. $|I|^2$ can be approximated by one-half of its envelope; when $k_x^2>>k_m^2$, the analytic expression of the envelope simplifies. The integral approximation for positive k_x is

$$\int_{k_x>k_b} K|I|^2 \, dk_x \approx (2k_m^2/\ell^2)\int K \ k_x^{-4} \ dk_x \ , \qquad \text{(AII.11)}$$

and a similar one holds for negative k_x.

<u>Small Mode Numbers m</u>

Although the function $|I|^2$ has essentially similar characteristics for all large integers $m = k_m\ell/\pi$, its character is somewhat modified when m is small. The modifications are particularly marked for m = 1 and 2; graphs of the functions of these cases are given in Fig. AII.2. For m = 2, there are no minor peaks between the major peaks. When m = 1, the major peaks are merged into a single broad peak centered on $k_x = 0$.

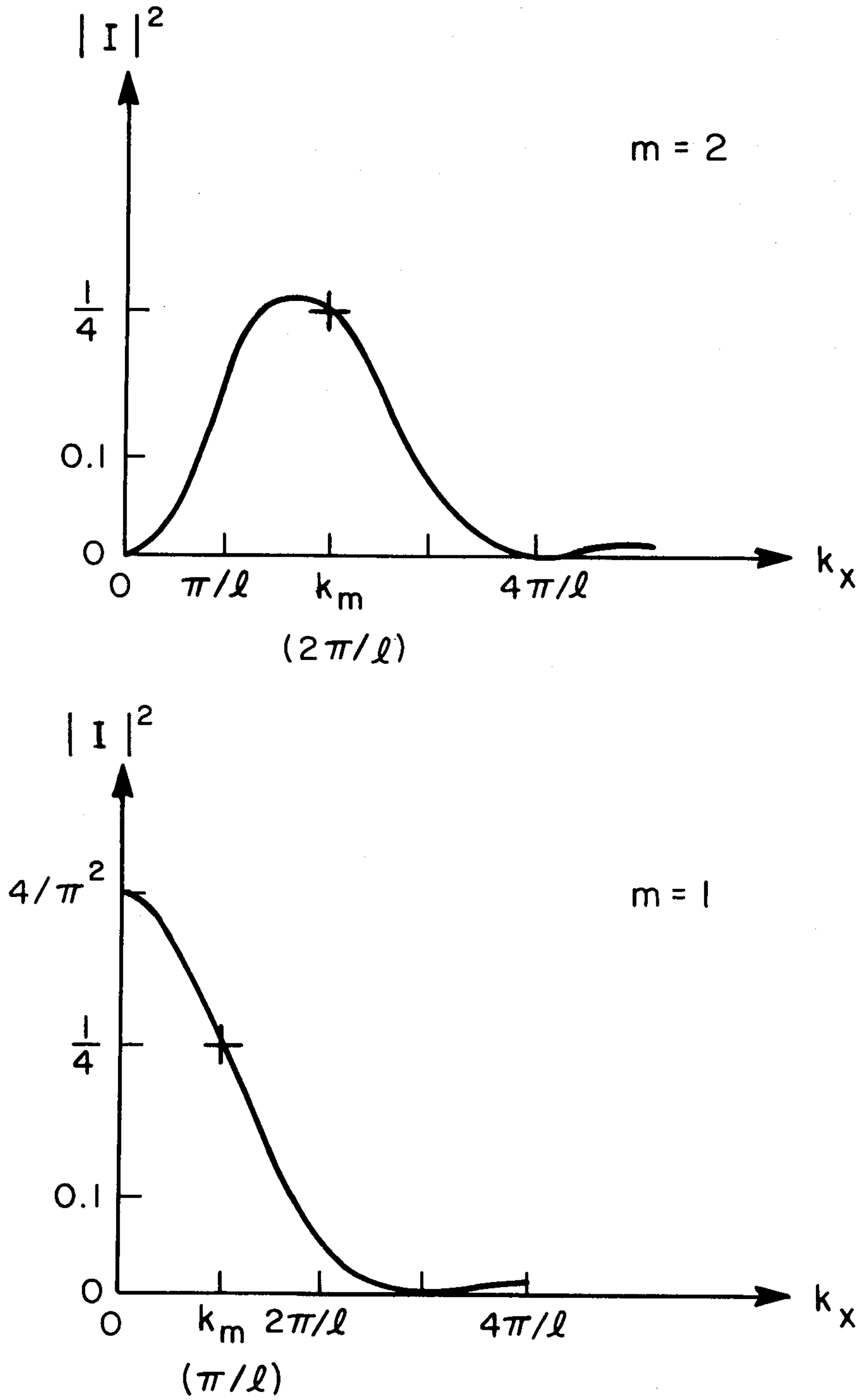

Figure AII.2.- Sketches of $|I|^2$ for m = 1, 2.

Some changes in the analytical approximations are required to reflect these differences. The first and fourth approximations (Eqs. AII.6, AII.11) are correct for all m. The second approximation is inapplicable to m = 1 and 2, because no such range of the variable exists. The third approximation (Eq. AII.10) is still correct if the lower limit of integration is set at zero. In explanation, note that Eq. AII.8 is correct for all m, as is readily shown by Parseval's theorem for Fourier transforms. Moreover, $|I|^2$ is small for $k_x > k_m$ for all m. These facts suffice to support the generalization of Eq. AII.10.

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